Computational Theory

Uncomputability

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Adapted from notes by Harry Lewis

Undecidability

Reading: Sipser §4.2, §5.1.

Motivation

- **Goal:** to find an explicit undecidable language
- By the Church-Turing thesis, such a language has a membership problem that cannot be solved by any kind of algorithm
- We know that such languages exist, by a counting argument:
  - Every decidable language is decided by a TM
  - There are only countably many TMs
  - There are uncountably many languages

  $\therefore$ Most languages are not decidable (or even Turing-recognizable)

Reminder: Three basic facts on decidability vs. recognizability

1. If L is decidable, then L is r.e.

   Proof:
   If M decides L, then a machine can recognize L by running M, and then going into an infinite loop if M would have halted in the q_{reject} state.

2. If L is recursive then so is $\overline{L}$.

   Proof:
   A machine can decide $\overline{L}$ by running M and then giving a "no" answer when M would give "yes" and *vice versa*.

3. L is recursive if and only if both L and $\overline{L}$ are r.e.

   Proof:

   ...

Is every Turing-recognizable set decidable?

This *would* be true if there were an algorithm to solve

> **The Acceptance Problem:**
> Given a TM M and an input w, does M accept input w?

Formally, $A_{\text{TM}} = \{\langle M, w\rangle : M \text{ accepts } w\}$.

Proposition: If A_{TM} is decidable, then every r.e. language is recursive.

"A_{TM} is the hardest r.e. language."

A_{TM} is said to be *r.e.-complete*

A simplifying detail: every string represents some TM

- Let Σ be the alphabet over which TMs are represented (that is, $\langle M \rangle \in \Sigma^*$ for any TM M)
- Let $w \in \Sigma^*$
- If $w = \langle M \rangle$ for some TM M then w represents M
- Otherwise w represents some fixed TM M_0 (say the simplest possible TM)

Thm: A_{TM} is not decidable

- Look at A_{TM} as a table answering every question:

	w_0	w_1	w_2	w_3
M_0	Y	N	N	Y
M_1	Y	Y	N	N
M_2	N	N	N	N
M_3	Y	Y	Y	Y

(WLOG assume every string w_i encodes a TM M_i)

- Entry matching (M_i, w_j) is Y iff M_i accepts w_j

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- Entry matching (M_i, w_j) is Y iff M_i accepts w_j
- If A_{TM} were decidable, then so would be the diagonal D and its complement.
  - $D = \{w_i : M_i \text{ accepts } w_i\}$
  - $\overline{D} = \{w_i : M_i \text{ does not accept } w_i\}$
- But $\overline{D}$ differs from every row, i.e., it differs from every r.e. language. $\Rightarrow\Leftarrow$

Unfolding the Diagonalization

- Suppose for contradiction that A_{TM} were recursive.
- Then there is a TM M^* that decides $\overline{D} = \{\langle M \rangle : M \text{ does not accept } \langle M \rangle\}$.
  - $M^*(\langle N \rangle)$ runs the decider for A_{TM} on $\langle N, \langle N \rangle\rangle$ and does the opposite.
- Run M^* on its own description $\langle M^* \rangle$.
- Does it accept?

$$M^* \text{ accepts } \langle M^* \rangle$$
$$\Leftrightarrow$$
$$\langle M^* \rangle \in \overline{D}$$
$$\Leftrightarrow$$
$$M^* \text{ does not accept } \langle M^* \rangle$$

- Contradiction!

Alan Mathison Turing

Alan Mathison Turing (1912–1954)

24 Years Old when he published *On computable numbers . . .*

Some More Undecidable Problems About TMs

- **The Halting Problem:** Given M and w, does M halt on input w?

  Proof:
  Suppose $HALT_{\text{TM}} = \{\langle M, w\rangle : M \text{ halts on } w\}$ were decided by some TM H.

  Then we could use H to decide A_{TM} as follows:

  On input $\langle M, w\rangle$,

  - Modify M so that whenever it is about to go into q_{reject}, it instead goes into an infinite loop. Call the resulting TM M'.
  - Run $H(\langle M', w\rangle)$ and do the same.

  Note that M' halts on w iff M accepts w, so this is indeed a decider for A_{TM}. $\Rightarrow\Leftarrow$.

Undecidable Problems, Continued

- For a certain fixed M_0:

  Given w, does M_0 halt on input w?

Undecidable Problems, Continued

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What about:

- For a fixed M_0 *and* a fixed w_0, does M_0 halt on input w_0?

Further Undecidable Problems

- Given M, does M halt on the empty string?

  Proof by reduction:

  Suppose M_1 decided $\{\langle M \rangle : M \text{ halts on } \varepsilon\}$.
  Then M_1 could be used to decide $HALT_{\text{TM}}$:

  Given $\langle M, w \rangle$,

  Construct $\langle M_w \rangle$, where M_w is a TM that writes w on the empty tape and then runs M.

  Then run M_1 on input $\langle M_w \rangle$

  M_1 accepts $\langle M_w \rangle \Leftrightarrow M_w$ halts on $\varepsilon \Leftrightarrow M$ halts on w

  But $HALT_{\text{TM}}$ is undecidable. $\Rightarrow\Leftarrow$

"Co-X"

- For any property X that a set might have, a set S is **co-X** iff $\overline{S}$ has property X.
- For example, a co-finite set of natural numbers is a set that is missing on a finite number of elements.
- A co-regular language is . . . ?
- A co-recursive language is . . . ?
- What about a co-CF language?
- Proved last time:
  - A language is recursive if and only if it is both r.e. and co-r.e.

Non-r.e. Languages

Theorem: The following languages are not r.e.:

- $\overline{A_{\text{TM}}} = \{\langle M, w\rangle : M \text{ does not accept } w\}$
- $\overline{HALT_{\text{TM}}} = \{\langle M, w\rangle : M \text{ does not halt on } w\}$
- $\overline{HALT^{\varepsilon}_{\text{TM}}} = \{\langle M\rangle : M \text{ does not halt on } \varepsilon\}$

Proof: If these languages were r.e. then A_{TM}, $HALT_{\text{TM}}$, and $HALT^{\varepsilon}_{\text{TM}}$ would be both r.e. and co-r.e. and hence recursive.

Is it possible to determine, given a TM M, whether M accepts a finite or infinite set?

- Let $A_{\text{infinite}} = \{\langle M \rangle : L(M) \text{ is infinite}\}$. Is A_{infinite} recursive?

Is it possible to determine, given a TM M, whether M accepts a finite or infinite set?

- Let $A_{\text{infinite}} = \{\langle M \rangle : L(M) \text{ is infinite}\}$. Is A_{infinite} recursive?
- Suppose M_2 decides A_{infinite}. To decide A_{TM}, given $\langle M \rangle$ and $\langle w \rangle$, construct $\langle M_w^* \rangle$ so that
  - If M accepts w then M_w^* accepts its input, regardless of what it is, and
  - If M does not accept w then M_w^* runs forever.
- Then run M_2 on input $\langle M_w^* \rangle$.
- $L(M_w^*)$ is either Σ^* (and therefore infinite) or $\emptyset$ (and therefore finite) depending on whether or not M accepts w.

Reduce A_{TM} to A_{infinite}; since A_{TM} is undecidable, so is A_{infinite}

Reductions and Rice's Theorem

Reading: Sipser Ch. 5

Formalizing the Notion of Reduction

- L_1 "reduces" to L_2 if we can use a "black box" for L_2 to build an algorithm for L_1.
- A function $f : \Sigma_1^* \to \Sigma_2^*$ is **computable** if there is a Turing machine that for every input $w \in \Sigma_1^*$, M halts with just $f(w)$ on its tape.
- A **(mapping) reduction** of $L_1 \subseteq \Sigma_1^*$ to $L_2 \subseteq \Sigma_2^*$ is a computable function

  $f : \Sigma_1^* \to \Sigma_2^*$ such that, for any $w \in \Sigma_1^*$,

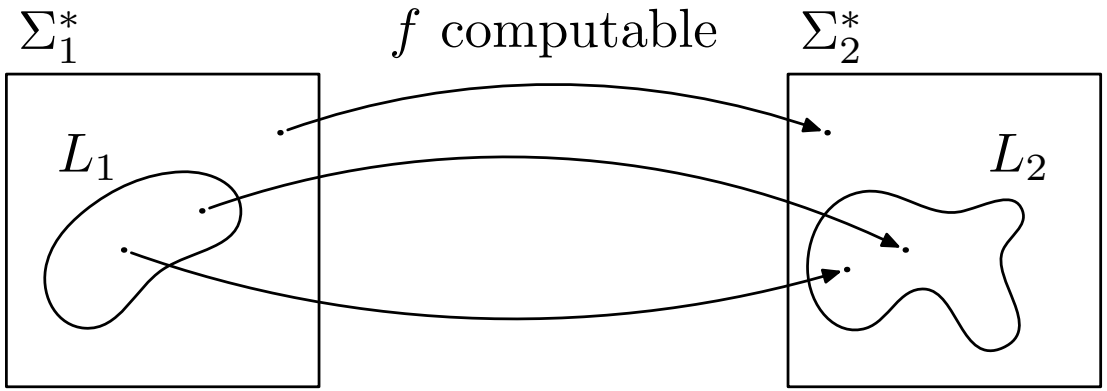
  $w \in L_1$ iff $f(w) \in L_2$

  We write $L_1 \leq_m L_2$.

Properties of reducibility

Lemma: If $L_1 \leq_m L_2$, then

- if L_2 is decidable (resp., r.e.), then so is L_1;
- if L_1 is undecidable (resp., non-r.e.), then so is L_2.



Examples of Reductions from Last Lecture

- For every Turing-recognizable L, $L \leq_m A_{\text{TM}}$.
- $A_{\text{TM}} \leq_m HALT_{\text{TM}}$.
- $HALT_{\text{TM}} \leq_m HALT^{\varepsilon}_{\text{TM}}$.

Rice's Theorem

Informally: **every** (nontrivial) property of Turing-recognizable languages is undecidable.

Rice's Theorem: Let $\mathcal{P}$ be any subset of the class of r.e. languages such that $\mathcal{P}$ and its complement are both nonempty. Then the language $L_{\mathcal{P}} = \{\langle M \rangle : L(M) \in \mathcal{P}\}$ is undecidable.

Thus, given a TM M, it is undecidable to tell if

- $L(M) = \emptyset$,
- $L(M)$ is regular,
- $|L(M)| = \infty$, etc.

Proof of Rice's Theorem

- We will reduce L_ε to $L_\mathcal{P}$.
- Suppose without loss of generality that $\emptyset \notin \mathcal{P}$.
- Pick any $L_0 \in \mathcal{P}$ and say $L_0 = L(M_0)$.
- Define $f(\langle M \rangle) = \langle M' \rangle$, where
  M' is a TM that on input w,
  - first simulates M on input ε
  - then simulates M_0 on input w
- **Claim:** f is a mapping reduction from L_ε to $L_\mathcal{P}$.
- Since L_ε is undecidable, so is $L_\mathcal{P}$.

Recursion Theory over $\mathcal{N}$

- We have presented this theory as a theory of languages
- Classically it is treated as a theory of sets of numbers
- The two are equivalent since strings can be converted to numbers (treating strings as numerals, for example) and v.v.
- So it makes sense to say "The set of primes is recursive"

Recursive functions

- A function f is recursive if f is computable
- e.g. if there is a TM that always leaves $f(w)$ on the tape when started with input w
- Similarly we can speak of a recursive function from numbers to numbers
- Thm: A set is r.e. iff it is the range of a recursive function or is $\emptyset$

Undecidable Problems and Unprovable Theorems

Reading: Sipser Ch. 5

Unsolvability of Derivability in General Grammars

- **Theorem:** There is no algorithm to determine, given any grammar G and any string w, whether $w \in L(G)$.

  Proof: Suppose there were such a decision procedure.

  Then we could use it to solve the halting problem:

  > Given M and w, to determine if M halts on input w, construct a grammar G such that $L(M) = L(G)$ and determine if $w \in L(G)$.

  Since the halting problem is unsolvable, so is this problem.

- There is a particular grammar G_0 for which this problem is unsolvable: namely, the grammar for the universal TM.

Two-Counter Machines

- A counter machine can add and subtract 1 from its registers and check if they are zero
- **Theorem:** The halting problem is unsolvable even for 2-counter machines.
- **Proof:**
  1. One TM tape to two pushdown stores
  2. One pushdown store to two counters
  3. Four counters to two counters

An Undecidable Problem about Context Free Grammars

Theorem: It is undecidable to determine, given CFGs G_1 and G_2, whether $L(G_1) \cap L(G_2) = \emptyset$.

Proof: Reduction from $\{\langle G, w\rangle : G$ is a general grammar generating $w\}$

- Given $\langle G, w\rangle$, we can construct grammars G_1, G_2 such that:

  $L(G_1) = \{C_1 \# D_1^R \# C_2 \# D_2^R \# \cdots \# C_n \# D_n^R$:
  $n \geq 1$, and for each i, $C_i \Rightarrow_G D_i\}$

  $L(G_2) = \{S \# C_2^R \# C_2 \# C_3^R \# \cdots \# C_n^R \# C_n \# w^R$:
  $n \geq 1$, and the C_i are arbitrary strings$\}$

- G_1 generates pairs of unrelated one-step derivations
- G_2 has S and w at beginning and end, and in between pairs match (even positions reversed)

Intersection of CFLs, continued

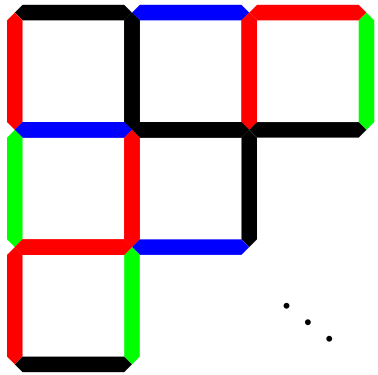
- Any string in $L(G_1)$ or $L(G_2)$ has an odd number of #s
- Any string in $L(G_1) \cap L(G_2)$ is a derivation of w in G (every other intermediate string is reversed)
- So $L(G_1) \cap L(G_2)$ is nonempty iff w is derivable in G
- So $\langle G, w \rangle \mapsto \langle G_1, G_2 \rangle$ is a reduction from general grammar derivability to $\{\langle G_1, G_2 \rangle : L(G_1) \cap L(G_2) \neq \emptyset\}$.

Verifying computations is easier than carrying them out!

Tiling

Tiling: Given a finite set of patterns for square tiles:

Is it possible to tile the whole plane with tiles of these patterns in such a way that the abutting edges match?



Theorem: Tiling is undecidable.

Tiling, continued

Variant of tiling: **fix** the tile at the origin and ask whether the first quadrant can be tiled (easier to show undecidability).

Proof by reduction from $\overline{L_\varepsilon}$:

- $\langle M \rangle \overset{f}{\mapsto}$ sets of tiles so that:
  M does not halt on $\varepsilon \Leftrightarrow f(\langle M \rangle)$ tiles the first quadrant.
- View computation of M as "tableau", filling first quadrant with elements of $C = Q \cup \Gamma$, each row being a configuration of M.
- Computation valid iff every 2×3 window consistent with transition function of M (and bottom row is correct initial configuration).
- Each tile represents a 2×3 window of tableau. Edge colors force consistency with neighbors on overlap.

Diophantine Equations

These are equations like

$$x^3y^3 + 13xyz = 4u^2 - 22$$

The coefficients and the exponents have to be *integers*. (No variables in the exponents!)

The question is whether the equation can be satisfied (made true) by substituting integers for the variables—this is known as Hilbert's 10th problem.

Diophantus of Alexandria (200–284 AD)

- "God gave him his boyhood one-sixth of his life, One twelfth more as youth while whiskers grew rife; And then yet one-seventh ere marriage begun; In five years there came a bouncing new son. Alas, the dead child of master and sage After attaining half the measure of his father's life chill fate took him. After consoling his fate by the science of numbers for four years, he ended his life."
- Other problems concerning triangular arrays, etc., gave rise to quadratic equations.
- Fermat's statement of his "Last Theorem" was in the margin of his copy of Diophantus.

"Hilbert's 10th Problem"

10. DETERMINATION OF THE SOLVABILITY OF A DIOPHANTINE EQUATION.

Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*

Thm: (Matiyasevish, 1970): Hilbert's 10th problem is unsolvable.

Relation to Gödel's Incompleteness Theorem

- Axiom systems for mathematics, e.g.
  - Peano arithmetic—attempt to capture properties of $\mathcal{N}$
    - E.g. mathematical induction:

      If $P[0]$

      and, for all n, $P[n] \Rightarrow P[n+1]$,

      then for all n, $P[n]$
  - Zermelo-Frankel-Choice set theory (ZFC)—enough for all of modern mathematics
- Proofs of theorems from these axioms defined by (simple) rules of mathematical logic.

The Decision Problem (for Mathematics)

- **Entscheidungsproblem** is German for "Decision Problem"
- **The** Decision Problem is the problem of determining whether a mathematical statement is provable
- **Proposition:** Set of provable theorems is Turing-recognizable.
- **Is it decidable?**

Undecidability of mathematics

- **Theorem [Church, Turing]:** Set of provable theorems is undecidable.

  Proof sketch:

  - Reduce from $HALT^{\varepsilon}_{\text{TM}}$
  - $\langle M \rangle \mapsto$ mathematical statement $\phi_M =$ "M halts on ε".
  - M halts on $\varepsilon \Rightarrow \phi_M$ has a proof.
  - M does not halt on $\varepsilon \Rightarrow \phi_M$ not true.

Incompleteness of Mathematics

- **Gödel's Incompleteness Theorem:** Some true statement is not provable.

  Proof sketch:

  - For every statement ϕ, either ϕ or $\neg\phi$ is true.
  - Suppose all true statements provable.

    $\Rightarrow$ For all statements ϕ, exactly one of ϕ and $\neg\phi$ is provable.

    $\Rightarrow$ Set of provable theorems is r.e. and co-r.e.

    $\Rightarrow$ Set of provable theorems recursive.
  - Contradiction.
- See Sipser Chapter 6 for more on this & other advanced topics on computability theory.