Computational Theory

Finite Automata and Regular Languages

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Adapted from notes by Harry Lewis

Finite Automata

Reading: Sipser §1.1 and §1.2.

Deterministic Finite Automata (DFAs)

Example: Home Stereo

- $P =$ power button (ON/OFF)
- $S =$ source button (CD/Radio/TV), only works when stereo is ON, but source remembered when stereo is OFF.
- Starts OFF, in CD mode
- **A computational problem**: does a given sequence of button presses $w \in \{P, S\}^*$ leave the system with the radio on?

Formal Definition of a DFA

- A DFA M is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$

  Q : Finite set of *states*
  Σ : Alphabet
  δ : "Transition function", $Q \times \Sigma \to Q$
  q_0 : Start state, $q_0 \in Q$
  F : Accept (or final) states, $F \subseteq Q$

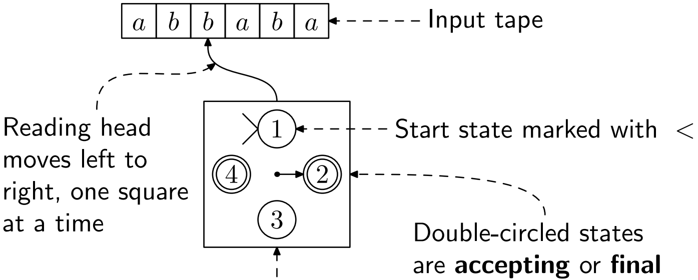
- If $\delta(p, \sigma) = q$,

  then if M is in state p and reads symbol $\sigma \in \Sigma$

  then M enters state q (while moving to next input symbol)

- Home Stereo example:

Another Visualization



Finite-state control changes state depending on:

- current state
- next symbol

Accepting Strings

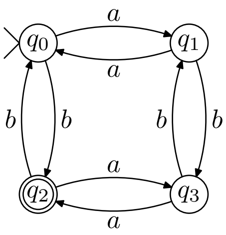
M accepts string X if

- After starting M in the start (initial) state with head on first square,
- when all of X has been read,
- M winds up in a final state.

Examples

- **Bounded Counting**: A DFA for

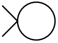
  $\{x : x$ has an even # of a's and an odd # of b's$\}$



Transition function δ:

	a	b
q_0	q_1	q_2
q_1	q_0	q_3
q_2	q_3	q_0
q_3	q_2	q_1

i.e. $\delta(q_0, a) = q_1$, etc.

 = start state    = final state

$Q = \{q_0, q_1, q_2, q_3\}$ $\Sigma = \{a, b\}$ $F = \{q_2\}$

Another Example, to work out together

- **Pattern Recognition**: A DFA that accepts $\{x : x$ has aab as a substring$\}$.

Formal Definition of Computation

$M = (Q, \Sigma, \delta, q_0, F)$ **accepts** $w = w_1 w_2 \cdots w_n \in \Sigma^*$ (where each $w_i \in \Sigma$) if there exist $r_0, \ldots, r_n \in Q$ such that

1. $r_0 = q_0$,
2. $\delta(r_i, w_{i+1}) = r_{i+1}$ for each $i = 0, \ldots, n-1$ and
3. $r_n \in F$.

The **language recognized** (or **accepted**) by M, denoted $L(M)$, is the set of all strings accepted by M.

Transition function on an entire string

More formal (not necessary for us, but notation sometimes useful):

- Inductively define $\delta^* : Q \times \Sigma^* \to Q$ by $\delta^*(q, \varepsilon) = q$, $\delta^*(q, w\sigma) = \delta(\delta^*(q, w), \sigma)$.
- Intuitively, $\delta^*(q, w) =$ "state reached after starting in q and reading the **string** w."
- M **accepts** w if $\delta^*(q_0, w) \in F$.

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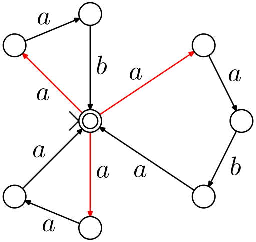
Determinism: Given M and w, the states $r_0, \ldots, r_n$ are uniquely determined. Or in other words, $\delta^*(q, w)$ is well defined for any q and w: There is precisely one state to which w "drives" M if it is started in a given state.

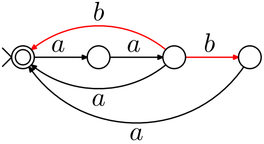
The impulse for nondeterminism

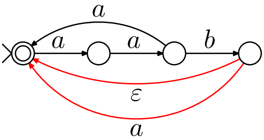
A language for which it is hard to design a DFA:

$$\{x_1 x_2 \cdots x_k : k \geq 0 \text{ and each } x_i \in \{aab, aaba, aaa\}\}$$

But it is easy to imagine a “device” to accept this language if there sometimes can be several possible transitions!







Nondeterministic Finite Automata

An **NFA** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q, Σ, q_0, F are as for DFAs
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \to \mathcal{P}(Q)$

When in state p reading symbol σ, can go to **any** state q in the **set** $\delta(p, \sigma)$.

- there may be more than one such q, or
- there may be none (in case $\delta(p, \sigma) = \emptyset$).

Can "jump" from p to any state in $\delta(p, \varepsilon)$ without moving the input head.

Computations by an NFA

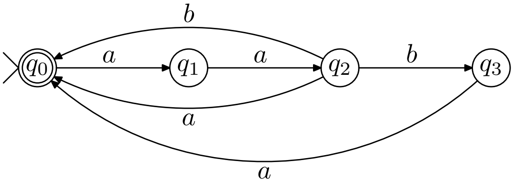
$N = (Q, \Sigma, \delta, q_0, F)$ **accepts** $w \in \Sigma^*$ if we can write $w = y_1 y_2 \ldots y_m$ where each $y_i \in \Sigma \cup \{\varepsilon\}$ and there exist $r_0, \ldots, r_m \in Q$ such that

1. $r_0 = q_0$,
2. $r_{i+1} \in \delta(r_i, y_{i+1})$ for each $i = 0, \ldots, m-1$, and
3. $r_m \in F$.

Nondeterminism: Given N and w, the states $r_0, \ldots, r_m$ are not necessarily determined.

Example of an NFA

N :



$N = (\{q_0, q_1, q_2, q_3\}, \{a, b\}, \delta, q_0, \{q_0\})$, where δ is given by:

	a	b	ε
q_0	$\{q_1\}$	$\emptyset$	$\emptyset$
q_1	$\{q_2\}$	$\emptyset$	$\emptyset$
q_2	$\{q_0\}$	$\{q_0, q_3\}$	$\emptyset$
q_3	$\{q_0\}$	$\emptyset$	$\emptyset$

Work out the tree of all possible computations on $aabaab$

How to simulate NFAs?

- NFA accepts w if there is at least one accepting computational path on input w.
- But the number of paths may grow exponentially with the length of w!
- Can exponential search be avoided?

NFAs and DFAs Closure Properties

Reading: Sipser §1.2.

NFAs vs. DFAs

NFAs *seem* more powerful than DFAs. Are they?

Theorem: For every NFA N, there exists a DFA M such that $L(M) = L(N)$.

Proof Outline: Given any NFA N, to construct a DFA M such that $L(M) = L(N)$:

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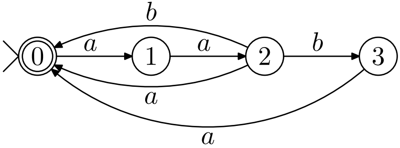
Proof Outline: Given any NFA N, to construct a DFA M such that $L(M) = L(N)$:

- Have the DFA keep track, at all times, of all possible states the NFA could be in after reading the same initial part of the input string.
- I.e., the **states** of M are **sets** of states of N, and $\delta^*_M(R, w)$ is the set of all states N could reach after reading w, starting from a state in R.

Example of the SUBSET CONSTRUCTION

NFA N for $\{x_1 x_2 \cdots x_k : k \geq 0$ and each $x_i \in \{aab, aaba, aaa\}\}$.

N :



N starts in state 0 so we will construct a DFA M starting in state $\{0\}$.

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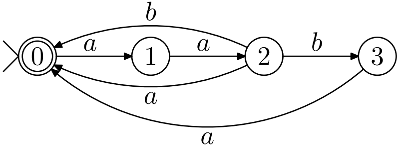
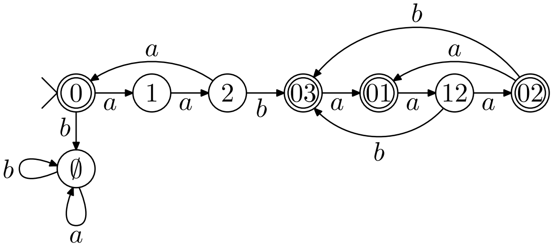


N starts in state 0 so we will construct a DFA M starting in state $\{0\}$. Here it is:



All other transitions are to the "dead state" $\emptyset$. The other states are unreachable, though technically must be defined. Final states are all those containing 0, the final state of N.

Formal Construction of DFA M from NFA $N = (Q, \Sigma, \delta, q_0, F)$

On the assumption that $\delta(p, \varepsilon) = \emptyset$ for all states p.
(i.e., we assume no ε-transitions, just to simplify things a bit)

$M = (Q', \Sigma, \delta', q_0', F')$ where

$$\begin{aligned} Q' &= \mathcal{P}(Q) \\ q_0' &= \{q_0\} \\ F' &= \{R \subseteq Q : R \cap F \neq \emptyset\} \text{ (that is, } R \in Q'\text{)} \\ \delta'(R, \sigma) &= \{q \in Q : q \in \delta(r, \sigma) \text{ for some } r \in R\} \\ &= \bigcup_{r \in R} \delta(r, \sigma) \end{aligned}$$

Proving that the construction works

Claim: For every string w, running M on input w ends in the state $\{q \in Q : \text{ some computation of } N \text{ on input } w \text{ ends in state } q\}$.

Pf: By induction on $|w|$.

Can be extended to work even for NFAs with ε-transitions.

“THE SUBSET CONSTRUCTION”

Closure Properties

Theorem: The class of regular languages is closed under:

- Union: $L_1 \cup L_2$
- Concatenation: $L_1 \circ L_2 = \{xy : x \in L_1$ and $y \in L_2\}$
- Kleene *: $L_1^* = \{x_1 x_2 \cdots x_k : k \geq 0$ and each $x_i \in L_1\}$
- Complement: $\overline{L_1}$
- Intersection: $L_1 \cap L_2$

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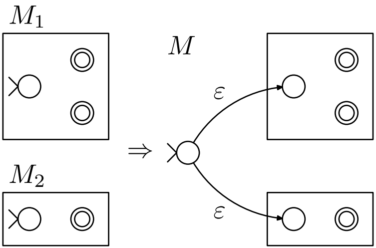
Union: If L_1 and L_2 are regular, then $L_1 \cup L_2$ is regular.



M has the states and transitions of M_1 and M_2 plus a new start state ε-transitioning to the old start states.

Concatenation, Kleene*, Complementation

Concatenation:
$L(M) = L(M_1) \circ L(M_2)$

Kleene*:
$L(M) = L(M_1)^*$

Complement:
$L(M) = \overline{L(M_1)}$

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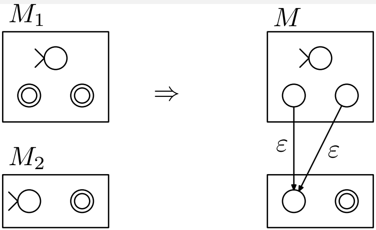

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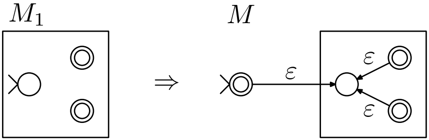
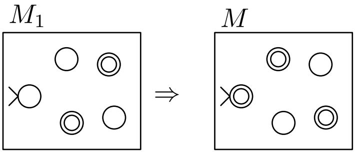


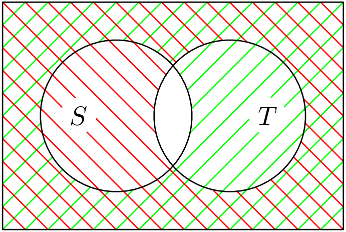
Complement:
$L(M) = \overline{L(M_1)}$

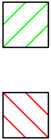


- Assume M is deterministic (or make it so)
- Invert final/ nonfinal states

Closure under intersection

Intersection: $S \cap T = \overline{\overline{S} \cup \overline{T}}$





Hence closure under union and complement implies closure under intersection.

A more constructive and direct proof of closure under intersection

Better way ("Cross Product Construction"):

From DFAs $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, construct $M = (Q, \Sigma, \delta, q_0, F)$:

$$
\begin{aligned}
Q &= Q_1 \times Q_2 \\
F &= F_1 \times F_2 \\
\delta(\langle r_1, r_2 \rangle, \sigma) &= \langle \delta_1(r_1, \sigma), \delta_2(r_2, \sigma) \rangle \\
q_0 &= \langle q_1, q_2 \rangle
\end{aligned}
$$

Then $L(M_1) \cap L(M_2) = L(M)$

Some Efficiency Considerations

The subset construction shows that any n-state NFA can be implemented as a 2^n-state DFA.

NFA States	DFA States
4	16
10	1024
100	2^{100}
1000	$2^{1000} \gg$ the number of particles in the universe

How to implement this construction on an ordinary digital computer?

NFA states

$1, \ldots, n$

DFA state bit vector

$$\begin{array}{cccccc} \boxed{0} & \boxed{1} & \boxed{1} & \boxed{0} & \boxed{\cdots} & \boxed{1} \\ 1 & 2 & & & & n \end{array}$$

Is this construction the best we can do?

Could there be a construction that always produces an n^2 state DFA for example?

Theorem: For every $n \geq 1$, there is a language L_n such that

1. There is an $(n+1)$-state NFA recognizing L_n.
2. There is no DFA recognizing L_n with fewer than 2^n states.

Conclusion: For finite automata, nondeterminism provides an **exponential savings** over determinism (in the worst case).

Proving that exponential blowup is sometimes unavoidable

(Could there be a construction that always produces an n^2 state DFA for example?)

Consider (for some fixed $n = 17$, say)

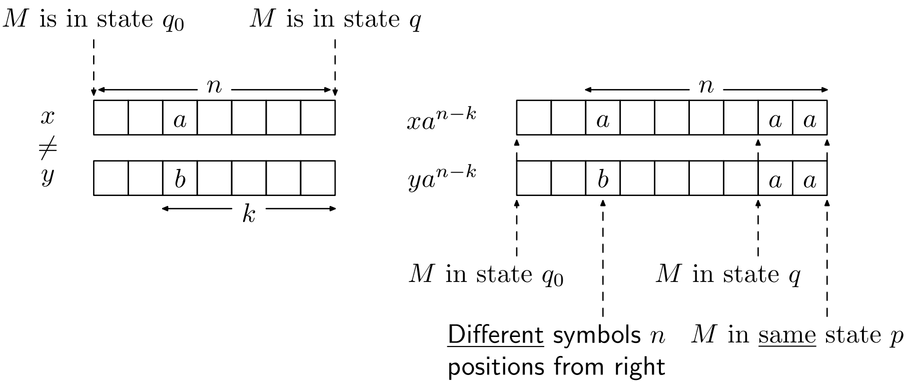
$$L_n = \{w \in \{a, b\}^* : \text{ the } n\text{th symbol from the right end of } w \text{ is an } a\}$$

- There is an $(n+1)$-state NFA that accepts L_n.
- There is no DFA that accepts L_n and has $< 2^n$ states

A "Fooling Argument"

- Suppose a DFA M has $< 2^n$ states, and $L(M) = L_n$
- There are 2^n strings of length n.
- By the pigeonhole principle, two such strings $x \neq y$ must drive M to the same state q.
- Suppose x and y differ at the k^{th} position from the right end (one has a, the other has b)
  $(k = 1, 2, \ldots,$ or $n)$
- Then M must treat xa^{n-k} and ya^{n-k} identically (accept both or reject both). These strings differ at position n from the right end.
- So $L(M) \neq L_n$, contradiction. QED.

Illustration of the fooling argument



- x and y are different strings
  (so there is a position k where one has a and the other has b)
- But both strings drive M from s to the same state q

What the argument proves

- This shows that the subset construction is within a factor of 2 of being optimal
- In fact it is optimal, i.e., as good as we can do in the worst case
- In many cases, the “generate-states-as-needed” method yields a DFA with $\ll 2^n$ states

  (e.g. if the NFA was deterministic to begin with!)

Regular Expressions

Reading: Sipser §1.3.

Regular Expressions

- Let $\Sigma = \{a, b\}$. The **regular expressions** over Σ are certain expressions formed using the symbols $\{a, b, (,), \varepsilon, \emptyset, \cup, \circ, {}^*\}$
- We use red for the strings under discussion (the **object language**) and **black** for the ordinary notation we are using for doing mathematics (the **metalanguage**).
- **Construction Rules** (= inductive/recursive definition):
  1. $a, b, \varepsilon, \emptyset$ are regular expressions
  2. If R_1 and R_2 are RE's, then so are $(R_1 \circ R_2)$, $(R_1 \cup R_2)$, and (R_1^*).
- Examples:
  - $(a \circ b)$
  - $((((a \circ (b^*)) \circ c) \cup ((b^*) \circ a))^*)$
  - $(\emptyset^*)$

What REs Do

- Regular expressions (which are strings) represent languages (which are sets of strings), via the function L:

$$
\begin{array}{lrcl}
(1) & L(a) & = & \{a\} \\
(2) & L(b) & = & \{b\} \\
(3) & L(\varepsilon) & = & \{\varepsilon\} \\
(4) & L(\emptyset) & = & \emptyset \\
(5) & L((R_1 \circ R_2)) & = & L(R_1) \circ L(R_2) \\
(6) & L((R_1 \cup R_2)) & = & L(R_1) \cup L(R_2) \\
(7) & L((R_1^*)) & = & L(R_1)^*
\end{array}
$$

- Example:

$$L(((a^*) \circ (b^*))) = \{a\}^* \circ \{b\}^*$$

- $L(\cdot)$ is called the **semantics** of the expression.

Syntactic Shorthand

- Drop the distinction between red and black, between object language and metalanguage
- Omit $\circ$ symbol and many parentheses
- Union and concatenation of languages are associative

  i.e., for any languages L_1, L_2, L_3:

  $(L_1 L_2) L_3 = L_1 (L_2 L_3)$ and $(L_1 \cup L_2) \cup L_3 = L_1 \cup (L_2 \cup L_3)$

  so we can write just $R_1 R_2 R_3$ and $R_1 \cup R_2 \cup R_3$

  For example, the following are all equivalent:

  $$((ab)c) \qquad (a(bc)) \qquad abc$$

- **Equivalent** means “same semantics, maybe different syntax”

More syntactic sugar

- By convention, * takes precedence over $\circ$, which takes precedence over $\cup$.

  So $a \cup bc^*$ is equivalent to $(a \cup (b \circ (c^*)))$
- Σ is shorthand for $a \cup b$ (or the analogous RE for whatever alphabet is in use).

Examples of Regular Languages

Strings ending in $a = \Sigma^* a$

Strings containing the substring $abaab =$?

Strings of even length $= (aa \cup ab \cup ba \cup bb)^*$

Strings with even # of a's

$$\begin{aligned} &= (b \cup ab^* a)^* \\ &= b^*(ab^* ab^*)^* \end{aligned}$$

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Strings with $\leq$ two a's $=$?

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Strings with $\leq$ two a's = ?

Strings of form $x_1 x_2 \ldots x_k$, $k \geq 0$, each $x_i \in \{aab, aaba, aaa\}$ = ?

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Strings with $\leq$ two a's $=$?

Strings of form $x_1 x_2 \ldots x_k, k \geq 0$, each $x_i \in \{aab, aaba, aaa\} =$?

Decimal numerals, no leading zeros
$= 0 \cup ((1 \cup \ldots \cup 9)(0 \cup \ldots \cup 9)^*)$

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All strings with an even # of a's *and* an even # of b's
$= (b \cup ab^* a)^* \cap (a \cup ba^* b)^*$ *but this isn't a regular expression*

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All strings with an even # of a's *and* an even # of b's
$= (b \cup ab^* a)^* \cap (a \cup ba^* b)^*$ *but this isn't a regular expression*
$= (aa \cup bb)^*((ab \cup ba)(aa \cup bb)^*(ab \cup ba)(aa \cup bb)^*)^*$

Equivalence of REs and FAs

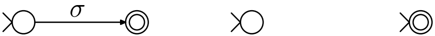
Recall: we call a language **regular** if there is a finite automaton that recognizes it.

Theorem: For every regular expression R, $L(R)$ is regular.

Proof (going back to hyper-formality for a moment):

Induct on the construction of regular expressions ("structural induction").

Base Case: R is a, b, ε, or $\emptyset$



accepts $\{\sigma\}$ accepts $\emptyset$ accepts $\{\varepsilon\}$

Equivalence of REs and FAs, continued

Inductive Step: If R_1 and R_2 are REs and $L(R_1)$ and $L(R_2)$ are regular (inductive hyp.), then so are:

$$\begin{aligned} L((R_1 \circ R_2)) &= L(R_1) \circ L(R_2) \\ L((R_1 \cup R_2)) &= L(R_1) \cup L(R_2) \\ L((R_1^*)) &= L(R_1)^* \end{aligned}$$

(By the closure properties of the regular languages).

Proof is **constructive** (actually produces the equivalent NFA, not just proves its existence).

Example conversion of a RE to a FA

$(a \cup \varepsilon)(aa \cup bb)^*$

The Other Direction

Theorem: For every regular language L, there is a regular expression R such that $L(R) = L$.

Proof:

Define **generalized NFAs** (GNFAs) (of interest only for this proof)

- Transitions labelled by regular expressions (rather than symbols).
- One start state q_{start} and only one accept state q_{accept}.
- Exactly one transition from q_i to q_j for every two states $q_i \neq q_{\text{accept}}$ and $q_j \neq q_{\text{start}}$ (including self-loops).

Steps toward the proof

Lemma: For every NFA N, there is an equivalent GNFA G.

- Add new start state, new accept state. Transitions?
- If multiple transitions between two states, combine. How?
- If no transition between two states, add one. With what transition?

Lemma: For every GNFA G, there is an equivalent RE R.

- By induction on the number of states k of G.
- **Base case:** $k = 2$. Set R to be the label of the transition from q_{start} to q_{accept}.

Ripping and repairing GNFAs to reduce the number of states

- **Inductive Hypothesis:** Suppose every GNFA G of k or fewer states has an equivalent RE (where $k \geq 2$).
- **Induction Step:** Given a $(k+1)$-state GNFA G, we will construct an equivalent k-state GNFA G'.

  Rip: Remove a state q_r (other than q_{start}, q_{accept}).

  Repair: For every two states $q_i \notin \{q_{\text{accept}}, q_r\}$, $q_j \notin \{q_{\text{start}}, q_r\}$, let $R_{i,j}$, $R_{i,r}$, $R_{r,r}$, $R_{r,j}$ be REs on transitions $q_i \to q_j$, $q_i \to q_r$, $q_r \to q_r$ and $q_r \to q_j$ in G, respectively,

  In G', put RE $R_{i,j} \cup R_{i,r}R^*_{r,r}R_{r,j}$ on transition $q_i \to q_j$.

  Argue that $L(G') = L(G)$, which is regular by IH.

Also **constructive**.

Example conversion of an NFA to a RE

An NFA accepting strings with an even number of a's and an even number of b's.

Countability

Reading: Sipser, “The Diagonalization Method,” pages 174–178 (from just before Definition 4.12 up to Corollary 4.18).

Examples of Regular Languages

- $\{w \in \{a, b\}^* : |w|$ even & every 3rd symbol is an $a\}$
- $\{w \in \{a, b\}^* :$ There are not 7 a's or 7 b's in a row$\}$
- $\{w \in \{a, b\}^* : w$ has both an even number of a's and an even number of b's$\}$
- $\{w : w$ is written using the ASCII character set and every substring delimited by spaces, punctuation marks, or the beginning or end of the string is in the American Heritage Dictionary$\}$

Questions about regular languages

Give $X =$ a regular expression, DFA, or NFA, how could you tell if:

- $x \in L(X)$, where x is some string?
- $L(X) = \emptyset$?
- $x \in L(X)$ but $x \notin L(Y)$?
- $L(X) = L(Y)$, where Y is another RE/FA?
- $L(X)$ is infinite?
- There are infinitely many strings that belong to both $L(X)$ and $L(Y)$?

Goal: Existence of Non-Regular Languages

Intuition:

- Every regular language can be described by a finite string (namely a regular expression).
- To specify an arbitrary language requires an infinite amount of information.
  - For example, an infinite sequence of bits would suffice.
  - Σ^* has a lexicographic ordering, and the i'th bit of an infinite sequence specifying a language would say whether or not the i'th string is in the language.

$\Rightarrow$ Some languages must not be regular.

How to formalize?

Countability

- A set S is **finite** if there is a bijection $\{1, \ldots, n\} \leftrightarrow S$ for some $n \geq 0$.
- **Countably infinite** if there is a bijection $f : \mathcal{N} \leftrightarrow S$

  This means that S can be "enumerated," i.e. listed as $\{s_0, s_1, s_2, \ldots\}$ where $s_i = f(i)$ for $i = 0, 1, 2, 3, \ldots$

  So $\mathcal{N}$ itself is countably infinite

  So is $\mathcal{Z}$ (integers) since $\mathcal{Z} = \{0, -1, 1, -2, 2, \ldots\}$

  Q: What is f?
- **Countable** if S is finite or countably infinite
- **Uncountable** if it is not countable

Facts about Infinite Sets

- **Proposition:** The union of 2 countably infinite sets is countably infinite.

  If $A = \{a_0, a_1, \ldots\}$, $B = \{b_0, b_1, \ldots\}$

  The $A \cup B = C = \{c_0, c_1, \ldots\}$

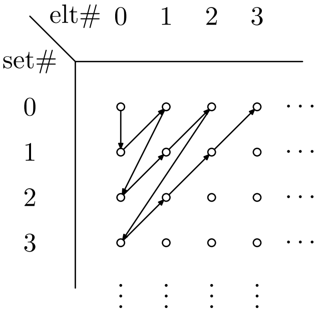
  $$\text{where } c_i = \begin{cases} a_{i/2} & \text{if } i \text{ is even} \\ b_{(i-1)/2} & \text{if } i \text{ is odd} \end{cases}$$

  Q: If we are being fussy, there is a small problem with this argument. What is it?

- **Proposition:** If there is a function $f : \mathcal{N} \rightarrow S$ that is onto S then S is countable.

Countable Unions of Countable Sets

- **Proposition:** The union of countably many countably infinite sets is countably infinite



$(0, 0)$,
$(0, 1), (1, 0)$,
$(0, 2), (1, 1), (2, 0)$,
$\ldots$

Each element is “reached” eventually in this ordering

Q: What is the bijection $\mathcal{N} \leftrightarrow \mathcal{N} \times \mathcal{N}$?

Are there uncountable sets? (Infinite but not countably infinite)

Theorem: $\mathcal{P}(\mathcal{N})$ is uncountable
(The set of all sets of natural numbers)

Proof by contradiction: (i.e. assume that $\mathcal{P}(\mathcal{N})$ is countable and show that this results in a contradiction)

- Suppose that $\mathcal{P}(\mathcal{N})$ were countable.
- There there is an enumeration of all subsets of $\mathcal{N}$ say $\mathcal{P}(\mathcal{N}) = \{S_0, S_1, \ldots\}$

Diagonalization

$$\begin{array}{c|cccccc}
 & j=0 & 1 & 2 & 3 & 4 & \cdots \\
S_i & & & & & & \\
\hline
S_0 & Y & N & N & Y & N & \cdots \\
S_1 & N & N & N & N & N & \cdots \\
S_2 & Y & Y & N & Y & Y & \cdots \\
S_3 & N & N & N & Y & N & \cdots \\
\vdots & & & & D & &
\end{array}$$

"Y" in row i, column j means $j \in S_i$

- Let $D = \{i \in \mathcal{N} : i \in S_i\}$ be the diagonal
- $D = YNNY \ldots = \{0, 3, \ldots\}$
- Let $\overline{D} = \mathcal{N} - D$ be its complement
- $\overline{D} = NYYN \ldots = \{1, 2, \ldots\}$
- **Claim:** $\overline{D}$ is omitted from the enumeration, contradicting the assumption that every set of natural numbers is one of the S_is.

  Pf: $\overline{D}$ is different from each row; they differ at the diagonal.

Cardinality of Languages

- An alphabet Σ is finite by definition
- **Proposition:** Σ^* is countably infinite
- So every language is either finite or countably infinite
- $\mathcal{P}(\Sigma^*)$ is uncountable, being the set of subsets of a countable infinite set.

  i.e. There are uncountably many languages over any alphabet

  Q: Even if $|\Sigma| = 1$?

Existence of Non-regular Languages

Theorem: For every alphabet Σ, there exists a non-regular language over Σ.

Proof:

- There are only countably many regular expressions over Σ.
  $\Rightarrow$ There are only countably many regular languages over Σ.
- There are uncountably many languages over Σ.
- Thus at least one language must be non-regular.

$\Rightarrow$ In fact, "almost all" languages must be non-regular.

Q: Could we do this proof using DFAs instead?

Q: Can we get our hands on an *explicit* non-regular language?

Non-Regular Languages

Reading: Sipser, §1.4.

Goal: Explicit Non-Regular Languages

It *appears* that a language such as

$$\begin{aligned} L &= \{x \in \Sigma^* : |x| = 2^n \text{ for some } n \geq 0\} \\ &= \{a, b, aa, ab, ba, bb, aaaa, \ldots, bbbb, aaaaaaaa, \ldots\} \end{aligned}$$

can't be regular because the "gaps" in the set of possible lengths become arbitrarily large, and no DFA could keep track of them.

But this isn't a proof!

Approach:

1. Prove some general property P of all regular languages.
2. Show that L does **not** have P.

Pumping Lemma (Basic Version)

If L is regular, then there is a number p (the **pumping length**) such that
every string $s \in L$ of length at least p
can be divided into $s = xyz$, where $y \neq \varepsilon$ and
for every $n \geq 0, xy^n z \in L$.

$n = 1$	x	y	z	
$n = 0$	x	z		
$n = 2$	x	y	y	z

$\ldots$

Pumping Lemma (Basic Version)

If L is regular, then there is a number p (the **pumping length**) such that every string $s \in L$ of length at least p can be divided into $s = xyz$, where $y \neq \varepsilon$ and for every $n \geq 0, xy^n z \in L$.

$n = 1$	x	y	z	
$n = 0$	x	z		
$n = 2$	x	y	y	z

$\ldots$

- Why is the part about p needed?
- Why is the part about $y \neq \varepsilon$ needed?

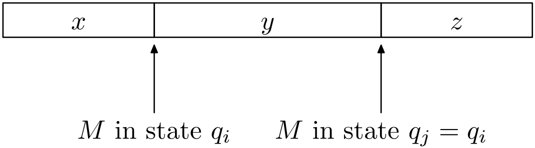
Proof of Pumping Lemma

(Another fooling argument)

- Since L is regular, there is a DFA M accepting L.
- Let $p = \#$ states in M.
- Suppose $s \in L$ has length $l \geq p$.
- M passed through a sequence of $l + 1 > p$ states while accepting s (including the first and last states): say, $q_0, \ldots, q_l$.
- Two of these states must be the same: say, $q_i = q_j$ where $i < j$

Pumping, continued

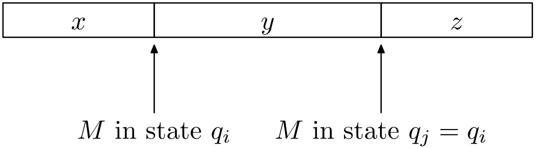
- Thus, we can break s into x, y, z where $y \neq \varepsilon$ (though x, z may equal ε):



- If more copies of y are inserted, M "can't tell the difference," i.e., the state entering y is the same as the state leaving it.
- So since $xyz \in L$, then $xy^n z \in L$ for all n.

Pumping, continued

- Thus, we can break s into x, y, z where $y \neq \varepsilon$ (though x, z may equal ε):



- If more copies of y are inserted, M "can't tell the difference," i.e., the state entering y is the same as the state leaving it.
- So since $xyz \in L$, then $xy^n z \in L$ for all n.

Proof also shows:

- We can take $p =$ # states in smallest DFA recognizing L.
- Can guarantee division $s = xyz$ satisfies $|xy| \leq p$ (or $|yz| \leq p$).

Pumping Lemma Example

- Consider

$$L = \{x : x \text{ has an even \# of } a\text{'s and an odd \# of } b\text{'s}\}$$

- Since L is regular, pumping lemma holds.
  (i.e., every sufficiently long string s in L is "pumpable")
- For example, if $s = aab$, we can write $x = \varepsilon$, $y = aa$, and $z = b$.

Pumping the even a's, odd b's language

Claim: L satisfies pumping lemma with pumping length $p = 4$.

Proof:

Pumping the even a's, odd b's language

Claim: L satisfies pumping lemma with pumping length $p = 4$.

Proof:

Consider any string s of length at least 4, and write $s = tu$ where $|t| = 4$

- Case 1: t has an even number of a's and an even number of b's. Then we can set $x = \varepsilon$, $y = t$, $z = u$.
- Case 2: t has 3 a's and 1 b. Then we can set $y = aa$.
- Case 3: t has 3 b's and 1 a. Then we can set $y = bb$.
- So L satisfies the pumping lemma with pumping length $p = 4$.

Q: Can the Pumping Lemma be used to prove that L is regular?
That is, does "Pumpable" $\Rightarrow$ Regular?

Use PL to Show Languages are *NOT* Regular

Claim: $L = \{a^n b^n : n \geq 0\} = \{\varepsilon, ab, aabb, aaabbb, \ldots\}$ is not regular.

Proof by contradiction:

- Suppose that L is regular.
- So L has some pumping length $p > 0$.
- Consider the string $s = a^p b^p$. Since $|s| = 2p > p$, we can write $s = xyz$ for some strings x, y, z as specified by the lemma.
- Claim: No matter how s is partitioned into xyz with $y \neq \varepsilon$, we have $xy^2z \notin L$.
- This violates the conclusion of the pumping lemma, so our assumption that L is regular must have been false.

Strings of exponential lengths are a nonregular language

Claim: $L = \{w : |w| = 2^n$ for some $n \geq 0\}$ is not regular.

Proof:

Strings of exponential lengths are a nonregular language

Claim: $L = \{w : |w| = 2^n$ for some $n \geq 0\}$ is not regular.

Proof:

- Suppose L satisfies the pumping lemma with pumping length p.
- Choose any string $s \in L$ of length greater than p, say $|s| = 2^n$. By pumping lemma, write $s = xyz$.
- Let $|y| = k$. Then $2^n - k, 2^n, 2^n + k, 2^n + 2 \cdot k, \ldots$ are all powers of two.
- This is impossible. QED.

"Regular Languages Can't Do Unbounded Counting"

Claim: $L = \{w : w$ has the same number of a's and b's$\}$ is not regular.

Proof #1:

- Use pumping lemma on $s = a^p b^p$ with $|xy| \leq p$ condition.

“Regular Languages Can’t Do Unbounded Counting”

Claim: $L = \{w : w$ has the same number of a’s and b’s$\}$ is not regular.

Proof #1:

- Use pumping lemma on $s = a^p b^p$ with $|xy| \leq p$ condition.

Proof #2:

- If L were regular, then $L \cap a^* b^*$ would also be regular.

Reprise on Regular Languages

Which of the following are necessarily regular?

- A finite language
- A union of a finite number of regular languages
- $\{x : x \in L_1 \text{ and } x \notin L_2\}$, L_1 and L_2 are both regular
- A subset of a regular language

What Happens During the Transformations?

- NFA $\rightarrow$ DFA
- DFA $\rightarrow$ Regular Expression
- Regular Expression $\rightarrow$ NFA

Minimizing DFAs

Many different DFAs accept the same language. But there is a smallest one—and we can find it!

- Let M be a DFA
- Say that states p, q of M are **distinguishable** if there is a string w such that exactly one of $\delta^*(p, w)$ and $\delta^*(q, w)$ is final.
- Start by dividing the states of M into two equivalence classes: the final and non-final states.

Minimizing DFAs, continued

- Break up the equivalence classes according to this rule: If p, q are in the same equivalence class but $\delta(p, \sigma)$ and $\delta(q, \sigma)$ are not equivalent for some $\sigma \in \Sigma$, then p and q must be separated into different equivalence classes.
- When all the states that must be separated have been found, form a new and finer equivalence relation.
- Repeat.
- How do we know that this process stops?