Computational Theory

Computability

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Adapted from notes by Harry Lewis

Turing Machines

Reading: Sipser §3.1.

Turing Machines

Objective: Define a computational model that is

- **General-purpose:**
  (as powerful as programming languages)
- **Formally Simple:**
  (we can prove what **cannot** be computed)

The Origins of Computer Science

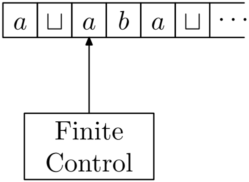
Alan Mathison Turing

> "On Computable Numbers, with an Application to the Entscheidungsproblem" 1936

CF also

- David Hilbert
  "Mathematical Problems" 1900
- Kurt Gödel
  "On Formally Undecidable Propositions . . . " 1931
- Alonzo Church
  "An Unsolvable Problem of Elementary Number Theory" 1936

The Basic Turing Machine



- Head can both read and write, and move in both directions
- Tape has unbounded length
- $\sqcup$ is the blank symbol. All but a finite number of tape squares are blank.

Formal Definition of a TM

A (deterministic) **Turing Machine (TM)** is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where:

- Q is a finite set of states, containing
  - the **start state** q_0
  - the **accept state** q_{accept}
  - the **reject state** q_{reject} $(\neq q_{\text{accept}})$
- Σ is the **input alphabet**
- Γ is the **tape alphabet**
  - Contains Σ
  - Contains “blank” symbol $\sqcup \in \Gamma - \Sigma$

The transition function

$$Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$$

- L and R are “move left” and “move right”
- $\delta(q, \sigma) = (q', \sigma', R)$
  - Rewrite σ as σ' in current cell
  - Switch from state q to state q'
  - And move right
- $\delta(q, \sigma) = (q', \sigma', L)$
  - Same, but move left
  - *Unless* at left end of tape, in which case stay put

Computation of TMs

- A **configuration** is uqv, where $q \in Q$, $u, v \in \Gamma^*$.
  - Tape contents $= uv$ followed by all blanks
  - State $= q$
  - Head on first symbol of v.
  - Equivalent to uqv', where $v' = v\sqcup$.
- Start configuration $= q_0 w$, where w is input.
- One step of computation:
  - $uq\sigma v$ yields $u\sigma' q' v$ if $\delta(q, \sigma) = (q', \sigma', R)$.
  - $u\tau q\sigma v$ yields $uq'\tau\sigma' v$ if $\delta(q, \sigma) = (q', \sigma', L)$.
  - $q\sigma v$ yields $q'\sigma' v$ if $\delta(q, \sigma) = (q', \sigma', L)$.
- If $q \in \{q_{\text{accept}}, q_{\text{reject}}\}$, computation halts.

TMs and Language Membership

- M **accepts** w if there is a sequence of configurations $C_1, \ldots, C_k$ such that
  1. $C_1 = q_0 w$.
  2. C_i yields C_{i+1} for each i.
  3. C_k is an accepting configuration (i.e. state of M is q_{accept}).
- $L(M) = \{w : M \text{ accepts } w\}$.
- L is **Turing-recognizable** if $L = L(M)$ for some TM M, i.e.
  - $w \in L \Rightarrow M$ halts on w in state q_{accept}.
  - $w \notin L \Rightarrow M$ halts on w in state q_{reject} OR M never halts (it “loops”).

Decidability, a.k.a. Recursiveness

- L is **(Turing-)decidable** if there is a TM M s.t.
  - $w \in L \Rightarrow M$ halts on w in state q_{accept}.
  - $w \notin L \Rightarrow M$ halts on w in state q_{reject}.
- Other common terminology
  - Recursive = decidable
  - Recursively enumerable (r.e.) = Turing-recognizable
  - Because of alternate characterizations as sets that can be defined via certain systems of recursive (self-referential) equations.

Example

Claim: $L = \{a^n b^n c^n : n \geq 0\}$ is decidable.

Questions

- Does every TM recognize some language?
- Does every TM decide some language?
- How many Turing-recognizable languages are there?
- How many decidable languages are there?

The Church-Turing Thesis

Reading: Sipser §3.2, §3.3.

"Computability"

- Defined in terms of Turing machines
- Computable = recursive/decidable (sets, functions, etc.)
- In fact an abstract, universal notion
- Many other computational models yield exactly the same classes of computable sets and functions
- Power of a model = what is computable using the model (extensional equivalence)
- Not programming convenience, speed (for now...), etc.
- All translations between models are **constructive**

TM Extensions That Do Not Increase Its Power

- TMs with a 2-way infinite tape, unbounded to left and right

$\cdots$	$\sqcup$	a	b	a	a	$\cdots$

Proof that TMs with 2-way infinite tapes are no more powerful than the 1-way infinite tape variety.

"Simulation." Convert any 2-way infinite TM into an equivalent 1-way infinite TM with a "two-track tape."

$\cdots$	c	b	a	$\sqcup$	b	a	$\sqcup$	b	a	a	$\cdots$
	-5	-4	-3	-2	-1	0	1	2	3	4	

Tape of 2-way infinite TM M

b
a

$= \langle b, a \rangle$

	-1	-2	-3	-4	-5	
\$	b	$\sqcup$	a	b	c	$\cdots$
	a	$\sqcup$	b	a	a	$\cdots$
	0	1	2	3	4	

Corresponding tape of 1-way infinite TM M'

Recall the Formal Definition of a TM:

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- Σ is the **input alphabet**
- Γ is the **tape alphabet**
  - Contains Σ
  - Contains "blank" symbol $\sqcup \in \Gamma - \Sigma$
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$ is the **transition function.**

Formalizing the Simulation of 2-way infinite tape TM

Formally, $\Gamma' = (\Gamma \times \Gamma) \cup \{\$\}$.

M' includes, for every state q of M, **two** states:

$\langle q, 1 \rangle \sim$ "q, but we are working on upper track"

$\langle q, 2 \rangle \sim$ "q, but we are working on lower track"

e.g. If $\delta_M(q, \sigma) = (q', \sigma', L)$ then $\delta_{M'}(\langle q, 1 \rangle, \langle \sigma, \tau \rangle) = (\langle q', 1 \rangle, \langle \sigma', \tau \rangle, R)$.

Also need transitions for:

- Lower track
- U-turn on hitting endmarker
- Formatting input into "2-tracks"

Describing Turing Machines

Formal Description

- 7-tuple or state diagram
- Most of the course so far

Implementation Description

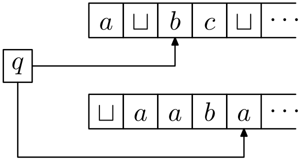
- Prose description of tape contents, head movements
- This lecture, some of next lecture, assignment 6

High-Level Description

- Does not refer to specific computational model
- Starting next time!

More extensions

- Adding **multiple tapes** does not increase power of TMs



2-tape TM

(Convention: First tape used for I/O, like standard TM; Second tape is available for scratch work)

Simulation of multiple tapes

- Simulate a k-tape TM by a one-tape TM whose tape is split (conceptually) into $2k$ **tracks**:
  - k tracks for tape symbols
  - k tracks for head position markers (one in each track)

<table>
<tr><td rowspan="4">$\$$</td><td>a</td><td>$\sqcup$</td><td>b</td><td>c</td><td>$\sqcup$</td><td>$\cdots$</td></tr>
<tr><td></td><td></td><td>$\uparrow$</td><td></td><td></td><td>$\cdots$</td></tr>
<tr><td>$\sqcup$</td><td>a</td><td>a</td><td>b</td><td>a</td><td>$\cdots$</td></tr>
<tr><td></td><td></td><td></td><td></td><td>$\uparrow$</td><td>$\cdots$</td></tr>
</table>

(Sipser does a different simulation.)

Simulation steps

- To simulate **one move** of the k-tape TM:

Simulation steps

- To simulate **one move** of the k-tape TM:
  - Start with the head on the left endmarker
  - Scan down the tape, remembering in the finite control the symbols "scanned" by the k heads
  - Scan back up the tape, revising each track in the vicinity of its head marker
  - Return the head to the left endmarker

Speed of the simulation

- Note that the "equivalence" in ability to compute functions or decide languages does not mean comparable **speed.**

  e.g. A standard TM can decide $L = \{w\#w : w \in \Sigma^*\}$ in time $\sim |w|^2$, but there is a **linear**-time 2-tape decider.

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- Note that the "equivalence" in ability to compute functions or decide languages does not mean comparable **speed.**

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- Let $T_M : \Sigma^* \to \mathcal{N}$ measure the amount of time a decider M uses on an input. That is, $T_M(w)$ is the number of steps TM M takes to halt on input w.

- General fact about multitape to single-tape slowdown:

  Theorem: If M is a multitape TM that takes time $T(w)$ when run on input w, then there is a 1-tape machine M' and a constant c such that M' simulates M and takes at most $cT(w)^2$ steps on input w.

Nondeterministic TMs

- Like TMs, but $\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$
- It mainly makes sense to think of NTMs as **recognizers**

$$L(M) = \{w : M \text{ has some accepting computation on input } w\}$$

Example: NTM to recognize
$\{w : w$ is a binary notation for a product of two integers $\geq 2\}$

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Example: NTM to recognize
$\{w : w$ is a binary notation for a product of two integers $\geq 2\}$

1. Write any binary numeral (except 0 or 1) [N.D.]
2. Write ⊔
3. Write any binary numeral (except 0 or 1) [N.D.]
4. Multiply
5. Compare product to the input; halt if they are equal, go into an infinite loop if not.

NTMs recognize the same languages as TMs

- Given a NTM M, we must construct a TM M' that determines, on input w, whether M has an accepting computation on input w.
- M' systematically tries
  - all one-step computations
  - all two-step computations
  - all three-step computations
  - ...

Enumerating computations

- There is a bounded number of k-step computations, for each k.
  (because for each configuration there is only a constant number of "next" configurations in one step)
- Ultimately M' either:
  - discovers an accepting computation of M, and accepts itself,
    or
  - searches forever, and does not halt

In More Detail

- Suppose that the maximum number of different transitions for a given (q, σ) is b.
- Number those transitions $1, \ldots, b$ (or less)
- Any computation of k steps is determined by a sequence of k numbers $\leq b$ (the "nondeterministic choices").
- How M' works: 3 tapes

$\#1$ | Original input to M $\sqcup$

$\#2$ | Simulated tape of M

$\#3$ | $1213 \sqcup \cdots$ Nondeterministic choices for M'

Simulating one step of M

- Each major phase of the simulation by M' is to simulate one finite computation by M, using tape #3 to resolve nondeterministic ambiguities.
- Between major phases, M'
  - erases tape #2 and copies tape #1 to tape #2
  - Replaces string in $\{1, \ldots, b\}^*$ on tape #3 with the lexicographically next string to generate the next set of nondeterministic choices to follow.
- **Claim:** $L(M') = L(M)$
- **Q:** Slowdown?

Equivalent Formalisms

Many other formalisms for computation are equivalent in power to the TM formalism:

- TMs with 2-dimensional tapes
- Random-access TMs
- General Grammars
- 2-stack PDAs, 2-counter machines
- Church's λ-calculus (μ-recursive functions)
- Markov algorithms
- Your favorite high-level programming language (C, Lisp, Java, . . .)
- . . .

General Grammars

- Like context-free grammars, except that if $u \to v$ is a rule, then u may be any string containing a nonterminal.
- So the rule $AXY \to AYX$ where $A, X, Y \in V$, "means" that the two-symbol substring XY can be replaced by YX whenever it appears with an A to its left.

Example of a General Grammar

A grammar to generate $\{a^n b^n c^n : n \geq 0\}$.

$\Sigma = \{a, b, c\} \quad V = \{A, B, C, A', B', C', S\}$

- A, B, C are "aliases" for the terminal symbols a, b, c.
- Only a single occurrence of A', B', or C' can be in the string being derived
- It "crawls" from right to left, transforming nonterminal symbols into terminals.

Rules for $a^n b^n c^n$

- $S \to ABCS \qquad S \to C' \qquad S \to \varepsilon$

  (Thus $S \overset{*}{\Rightarrow} (ABC)^n C'$ for any $n \geq 0$)

- $CA \to AC \qquad BA \to AB \qquad CB \to BC$

  (Any inversions of the proper order can be repaired)

- $CC' \to C'c \qquad CC' \to B'c$

  (The c-transformer can crawl to the left, and turn into a b-transformer)

- $BB' \to B'b \qquad BB' \to A'b$
- $AA' \to A'a \qquad A' \to \varepsilon$

The only way to get a string of **terminals** yields one of the form $a^n b^n c^n$.

Grammars and Turing Machines are Equivalent

Theorem: A language is generated by a grammar if and only if it is Turing-recognizable.

Proof:

1. L is generated by a grammar $\Rightarrow$ L is Turing-recognizable

   Pf: Let $L = L(G)$, G a grammar. To construct a NTM M such that $L(M) = L$, construct M so that

   M nondeterministically carries out a derivation $S = w_0 \Rightarrow_G w_1 \Rightarrow_G w_2 \Rightarrow_G \cdots$, checking each step to see if $w_i = w$.

L Turing-recognizable $\Rightarrow$ L is generated by a grammar.

2. L is recognized by a TM $M \Rightarrow L$ is generated by a grammar G

   Pf: Without loss of generality, we assume that if M halts having started on input w, right before halting it erases its tape.

   G will simulate a **backwards computation** by M. The intermediate strings will be configurations $\$uq\sigma v\$$.

Rules of the Grammar

- $S \to \$q_{\text{accept}}\$$
- If $\delta(q, \sigma) = (q', \sigma', R)$, then G has
  $\sigma' q' \to q\sigma$
  $\sigma' q' \$ \to q\$$, if $\sigma = \sqcup$
- If $\delta(q, \sigma) = (q', \sigma', L)$, then G has
  $q'\tau\sigma' \to \tau q\sigma$ for each $\tau \in \Sigma$
  $q'\tau\$ \to \tau q\sigma\$$, if $\sigma' = \sqcup$
  $\$q'\sigma' \to \$q\sigma$
- Finally, $\$ \to \varepsilon$ and, if q_0 is the start state of the TM, $q_0 \to \varepsilon$

Reduction of TMs to 2-CMs

A 2-counter machine (2-CM) has:

- A finite-state control
- Two counters, i.e., $C1$ and $C2$, which are registers containing integers ≥ 0 with only 3 operations:
  - Add 1 to $C1/C2$
  - Subtract 1 from $C1/C2$
  - Is $C1/C2 = 0$?

Theorem: For any TM, there is an equivalent 2-CM, in the sense that if you start the 2-CM with an encoding of the TM tape in its counters it will eventually halt with an encoding of what the TM computes.

Simulating a TM tape with 2 pushdown stores: Split the tape at the head position into two stacks

Moving TM head to left	$\equiv$	Pop from stack #1 Push onto stack #2
Moving TM head to right	$\equiv$	Pop from stack #2 Push onto stack #1
Change scanned symbol	$\equiv$	Change top of stack #1

(So 2-PDSs are as powerful as TMs)

Simulating One Stack with Two Counters: Think of the stack as a number in a base $= |\Sigma| + 1$

[Assume ≤ 9 stack symbols]

Pop the stack	$\equiv$	Divide by 10 and discard the remainder
Push a_9	$\equiv$	Multiply by 10 and add 9
Is stack top $= a_3$?	$\equiv$	Is counter mod $10 = 3$?

$\rightarrow$ All of these can be calculated using a second counter.

Simulating Four Counters With Two: $(p, q, r, s) \to 2^p 3^q 5^r 7^s$

Add 1 to $C1$	$\equiv$	$p \leftarrow p + 1$
	$\equiv$	Double $C1'$
Is $C3 \neq 0$?	$\equiv$	$r \neq 0$?
	$\equiv$	Does 5 divide $C1'$ evenly?
Subtract 1 from s	$\equiv$	Divide $C1'$ by 7

The Church-Turing Thesis

The equivalence of each to the others is a mathematical **theorem.**

That these **formal models** of algorithms capture our **intuitive notion** of algorithms is the **Church-Turing Thesis.**

(Church's thesis $=$ partial recursive functions, Turing's thesis $=$ Turing machines)

This is an extramathematical proposition, not subject to formal proof.

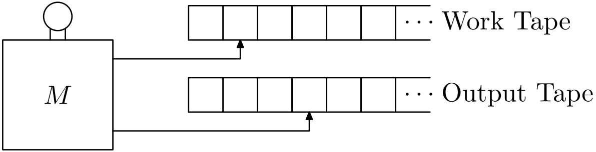
Decidability and a Universal Turing Machine

Reading: Sipser §4.1.

Another TM Variant: Enumerators

Def: A TM M ***enumerates*** a language L if M, when started from a blank tape, runs forever and "emits" all and only the strings in L.

(For example, by writing the string on a special tape and passing through a designated state.)



Recognizable $\equiv$ enumerable

Theorem: L is Turing-recognizable iff L is enumerated by some TM.

Proof:

($\Rightarrow$) Suppose $L(M) = L$. We want to construct a TM M' that enumerates L.

M' **dovetails** all of the computations by M:

1. Do 1 step of M's computation on w_0
2. Do 2 steps of M on w_0 and w_1
3. Do 3 steps on each of w_0, w_1, w_2

where $w_0, w_1, \ldots =$ lexicographic enumeration of Σ^*.

Outputting any strings w_i whose computations have accepted.

Recognizable $\equiv$ enumerable, finis

($\Leftarrow$) Conversely, suppose M enumerates L. We want to show that L is RE.

Given w, run M on the blank tape. Every time M passes through state q (the "enumeration state") pause to see if w is on the output tape and halt if it is.

The language **recognized** by this algorithm is exactly the language **enumerated** by M. QED.

- The Turing-decidable sets are usually called **recursive** because they can be computed using certain systems of recursive equations, rather than via TMs.
- The Turing-recognizable sets are usually called **recursively enumerable**, i.e., "computably enumerable."

Enumerable in order $\equiv$ decidability

Theorem: L is decidable iff L is enumerable in lexicographic order.

(lexicographic order has shorter strings before longer, and alphabetic order among strings of the same length)

Proof of $\Rightarrow$: If L is decidable, then to enumerate L in order, generate all of Σ^* in order and test each string for membership in L, enumerating those that are members.

Almost proof of $\Leftarrow$: to test if $w \in L$, enumerate L and wait until either w or a lexically later string is enumerated. ????

Decidability

- Recall that a language $L \subseteq \Sigma^*$ is decidable if there is a TM that always halts when started on an input in Σ^*, in either q_{accept} if $w \in L$ or q_{reject} if $w \notin L$.
- **Proposition:** Every regular language is decidable.

  Proof: (By "coding" a DFA as a TM.)

Asking questions about arbitrary finite automata

- **Q:** What if the DFA D is part of the input? That is, can we design a single TM that, given two inputs, D and w, decides whether D accepts w?
  - The TM needs to use a fixed alphabet & state set for all inputs D, w.

  Q: How to represent $D = (Q, \Sigma_D, \delta, q_0, F)$ and w?
  List each component of the 5-tuple, separated by $|$'s.
  - Represent elements of Q as binary strings over $\{0, 1\}$, seperated by ,'s.
  - Represent elements of Σ_D as binary strings over $\{0, 1\}$, seperated by ,'s.
  - Represent $\delta : Q \times \Sigma_D \to Q$ as a sequence of triples (q, σ, q'), separated by ,'s, etc.

  We denote the encoding of D and w as $\langle D, w \rangle$.

A "Universal" algorithm for deciding regular languages

- **Proposition:** $A_{\text{DFA}} = \{\langle D, w\rangle : D$ a DFA that accepts $w\}$ is decidable.

  Proof sketch:
  - First check that input is of proper form.
  - Then simulate D on w. Implementation on a multitape TM:
    - Tape 2: String w with head at current position (or to be precise, its representation).
    - Tape 3: Current state q of D (i.e., its representation).
- Could work with other encodings, e.g., transition function as a matrix rather than list of triples.

Representation independence

- **General point:** Notions of computability (e.g. decidability and recognizability) are independent of data representation.
  - A TM can convert any reasonable encoding to any other reasonable encoding.
  - We will use $\langle \cdot \rangle$ to mean "any reasonable encoding".
  - We will revisit representation issues when we discuss computational **speed**.
  - For the moment we are interested only in whether problems are decidable, undecidable, recognizable, etc., so we can be content knowing that there is **some** representation on which an algorithm could work.

Describing Turing Machines

Formal Description

- 7-tuple or state diagram
- Most of the course so far

Implementation Description

- Prose description of tape contents, head movements
- Previous lecture and today's lecture so far

High-Level Description

- Does not refer to specific computational model, data representation
- From now on!

More Decidable Problems

- $\{\langle R, w\rangle : R$ is a regular expression that generates $w\}$.
- $\{\langle X\rangle : X$ is a DFA/NFA/RE such that $L(X) = \emptyset\}$.
- $\{\langle X\rangle : X$ is a DFA/NFA/RE such that $|L(X)| = \infty\}$.
- $\{\langle M, w\rangle : M$ is a PDA that accepts $w\}$.
- Every context-free language.

A Universal Turing machine

- **Theorem:** There is a Turing machine U, such that when U is given $\langle M, w\rangle$ for any TM M and w, U produces the same result (accept/reject/loop) as running M on w.

  Proof: Initially,

  - First tape contains $\langle M\rangle$, including in particular its transition function δ_M.
  - Second tape contains $\langle w\rangle$.
  - Third tape contains $\langle q_{\text{start}}\rangle$.
  - Simulate steps of M by multiple steps of U.

  (Brief return to implementation description.)

  $\Rightarrow$ Turing machines can be "programmed".

Consequences of the existence of Universal Turing Machines

- **Corollary:** $A_{\mathsf{TM}} = \{\langle M, w\rangle : M \text{ accepts } w\}$ is Turing-recognizable (r.e.).
- **Corollary:** $HALT_{\mathsf{TM}} = \{\langle M, w\rangle : M \text{ eventually halts on } w\}$ ("The Halting Problem") is Turing-recognizable.
- **Corollary:** "The Turing Machines that halt on some input are an r.e. set" (What does this mean?)
- **Q:** What about $\{\langle M, w, n\rangle : M \text{ halts on } w \text{ in at most } n \text{ steps}\}$?
- **Q:** Are these sets decidable?
- **Q:** Are there undecidable languages?

Three basic facts on decidability vs. recognizability

1. If L is recursive, then L is r.e.

   Proof:

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   If M decides L, then a machine can recognize L by running M, and then going into an infinite loop if M would have halted in the q_{reject} state.

2. If L is recursive then so is $\overline{L}$.

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1. If L is recursive, then L is r.e.

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2. If L is recursive then so is $\overline{L}$.

   Proof:

   A machine can decide $\overline{L}$ by running M and then giving a "no" answer when M would give "yes" and *vice versa*.

3. L is recursive if and only if both L and $\overline{L}$ are r.e.

   Proof:

   ...