Computational Theory

Computational Complexity

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Fall 2017

Adapted from notes by Harry Lewis

Computational Complexity

Reading: Sipser §7.1, §7.2.

Objective of Complexity Theory

- To move the focus:
  - from what it is possible in principle to compute
  - to what is feasible to compute given “reasonable” resources
- For us the principle “resource” is time, though it could also be memory (“space”) or hardware (switches)

What is the "speed" of an algorithm?

Def: A TM M has **running time** $t : \mathcal{N} \rightarrow \mathcal{N}$ iff for all n, $t(n)$ is the maximum number of steps taken by M over *all* inputs of length n.

- $\rightarrow$ implies that M halts on every input
- $\rightarrow$ in particular, every decision procedure has a running time
- $\rightarrow$ time used as a function of size n
- $\rightarrow$ worst-case analysis

Example running times

Running times are generally increasing functions of n

$t(n) = 4n$

$t(n) = 2n \cdot \lceil \log n \rceil$

$\lceil x \rceil =$ least integer $\geq x$ (running times must be integers)

$t(n) = 17n^2 + 33$

$t(n) = 2^n + n$

$t(n) = 2^{2^n}$

"Table lookup" provides a speedup for finitely many inputs

Claim: For every decidable language L and every constant k, there is a TM M that decides L with running time $t(n) = n$ for all $n \leq k$.

Proof:

"Table lookup" provides a speedup for finitely many inputs

Claim: For every decidable language L and every constant k, there is a TM M that decides L with running time $t(n) = n$ for all $n \leq k$.

Proof:

- Answers to any finite number of "questions" can be built into the finite control of a TM, so that it takes no more time to answer these questions than the time needed to read the input
- So any **finite** language can be decided in the time needed to read the input.
- (Though size of TM grows in proportion to the number of hard-wired questions!)

$\Rightarrow$ study behavior only of Turing machines M deciding infinite languages, and only by analyzing the running time $t(n)$ as $n \to \infty$.

Why bother measuring TM time, when TMs are so miserably inefficient?

- **Answer:** Within limits, multitape TMs are a reasonable model for measuring computational speed.
- The trick is to specify the right amount of "slop" when stating that two algorithms are "roughly equivalent".
- Even coarse distinctions can be very informative.

Complexity Classes

- **Def:** Let $t : \mathcal{N} \to \mathcal{R}^+$. Then TIME$(t)$ is the class of languages L that can be decided by some **multitape** TM with running time $\leq t(n)$.

  e.g. TIME$(10^{10} \cdot n)$, TIME$(n \cdot 2^n)$

  $\mathcal{R}^+ =$ positive real numbers

- **Q:** Is it true that with more time you can solve more problems?
  i.e., if $g(n) < f(n)$ for all n, is TIME$(g) \subsetneq$ TIME(f)?

- **A:** Not exactly . . .

Linear Speedup Theorem

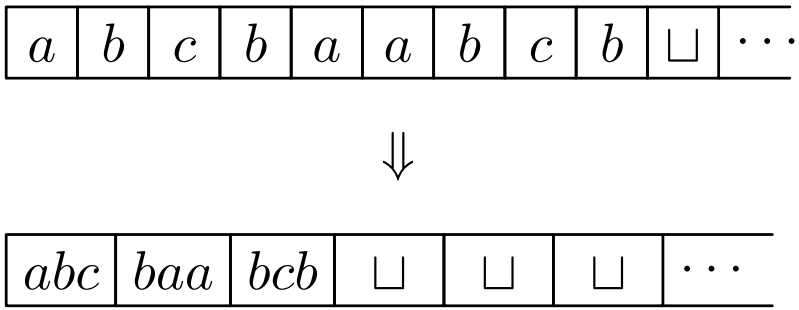
Let $t : \mathcal{N} \to \mathcal{R}^+$ be any function s.t. $t(n) \geq n$ and $0 < \varepsilon < 1$, then for every $L \in \text{TIME}(t)$, we also have $L \in \text{TIME}(\varepsilon \cdot t(n) + n)$

- $n =$ time to read input
- Note implied quantification:

  ($\forall$ TM M)($\forall \varepsilon > 0$)($\exists$ TM M') M' is equivalent to M but runs in fraction ε of the time.
- "Given any TM we can make it run, say, 1,000,000 times faster on all inputs."

Proof of Linear Speedup

- Let M be a TM deciding L in time T.
- A new, faster machine M':
  1. Copies its input to a second tape, in compressed form.



     - (Compression factor $= 3$ in this example—actual value TBD at end of proof)
  2. Moves head to beginning of compressed input.
  3. Simulates the operation of M treating all tapes as compressed versions of M's tapes.

Analysis of linear speedup

- Let the "compression factor" be c ($c = 3$ here), and let n be the length of the input.
- Running time of M':
  1. n steps.
  2. $\lceil n/c \rceil$ steps
     - $\lceil x \rceil =$ smallest integer $\geq x$
  3. takes ?? steps.

How long does the simulation (3) take?

- M' remembers in its finite control which of the c "subcells" M is scanning.

- M' keeps simulating c steps of M by 8 steps of M':
  1. Look at current cell on either side.
     (4 steps to read $3c$ symbols)
  2. Figure out the next c steps of M.
     (can't depend on anything outside these $3c$ subcells)
  3. Update these 3 cells and reposition the head.
     (4 steps)

End of simulation analysis

- It must do this $\lceil t(n)/c \rceil$ times, for a total of $8 \cdot \lceil t(n)/c \rceil$ steps.
- Total of $\leq (10/c) \cdot t(n) + n$ steps of M' for sufficiently large n.
- If c is chosen so that $c \geq 10/\varepsilon$ then M' runs in time $\varepsilon \cdot t(n) + n$.

Implications/Rationalizations of Linear Speedup

- "Throwing hardware at a problem" can speed up any algorithm by any desired constant factor
- E.g. moving from 8 bit $\rightarrow$ 16 bit $\rightarrow$ 32 bit $\rightarrow$ 64 bit parallelism
- Our theory does not "charge" for huge capital expenditures to build big machines, since they can be used for infinitely many problems of unbounded size
- This complexity theory is too weak to be sensitive to multiplicative constants—so we study **growth rate**

Growth Rates of Functions

- We need a way to compare functions according to how **fast** they increase, not just how **large** their values are.
- Intuitively, $f(n) = n^2$ grows faster than $g(n) = 10^{10} \cdot n$, even though for many values of n, $g(n) > f(n)$.
- **Def:** For $f : \mathcal{N} \to \mathcal{R}^+$, $g : \mathcal{N} \to \mathcal{R}^+$, we write $g = \mathcal{O}(f)$ if there exists $c, n_0 \in \mathcal{N}$ such that $g(n) \leq c \cdot f(n)$ for all $n \geq n_0$.
  - Binary relation: we could write $g = \mathcal{O}(f)$ as $g \preccurlyeq f$.
  - "If f is scaled up uniformly, it will be above g at all but finitely many points."
  - "g grows no faster than f."
  - Also write $f = \Omega(g)$.

Examples of Big-$\mathcal{O}$ notation

- If $f(n) = n^2$ and $g(n) = 10^{10} \cdot n$

  $g = \mathcal{O}(f)$ since $g(n) \leq 10^{10} \cdot f(n)$ for all $n \geq 0$

  where $c = 10^{10}$ and $n_0 = 0$

- Usually we would write: "$10^{10} \cdot n = \mathcal{O}(n^2)$"

  i.e. use an expression to name a function

- By Linear Speedup Theorem, TIME(t) is the class of languages L that can be decided by some multitape TM with running time $\mathcal{O}(t(n))$ (provided $t(n) \geq 1.01n$)

Examples

- $10^{10} \cdot n = \mathcal{O}(n^2)$.
- $1764 = \mathcal{O}(1)$.

  1: The constant function $1(n) = 1$ for all n.
- $n^3 \neq \mathcal{O}(n^2)$.
- Time $\mathcal{O}(n^k)$ for fixed k is considered "fast" ("polynomial time")
- Time $\Omega(k^n)$ is considered "slow" ("exponential time")
- Does this really make sense?

More Relations

- **Def:** We say that $g = o(f)$ iff for every $\varepsilon > 0$, $\exists n_0$ such that $g(n) \leq \varepsilon \cdot f(n)$ for all $n \geq n_0$.
  - Equivalently, $\lim_{n \to \infty} g(n)/f(n) = 0$
  - "g grows more slowly than f."
  - Also write $f = \omega(g)$.
- **Def:** We say that $f = \Theta(g)$ iff $f = \mathcal{O}(g)$ and $g = \mathcal{O}(f)$.
  - "g grows at the same rate as f"
  - An equivalence relation between functions.
  - The equivalence classes are called **growth rates**.
  - Because of linear speed up, TIME(t) is really the union of all growth rate classes $\preccurlyeq \Theta(t)$.

More Examples

- **Polynomials (of degree d):**
  $f(n) = a_d n^d + a_{d-1} n^{d-1} + \cdots + a_1 n + a_0$, where $a_d > 0$.
  - $f(n) = \mathcal{O}(n^c)$ for $c \geq d$.
  - $f(n) = \Theta(n^d)$
    - "If f is a polynomial, then lower order terms don't matter to the growth rate of f"
  - $f(n) = o(n^c)$ for $c > d$.
  - $f(n) = n^{\mathcal{O}(1)}$.

More Examples

- **Exponential Functions:** $g(n) = 2^{n^{\Theta(1)}}$.
  - Then $f = o(g)$ for any polynomial f.
  - $2^{n^\alpha} = o(2^{n^\beta})$ if $\alpha < \beta$.
- What about $n^{\lg n} = 2^{\lg^2 n}$?

  Here $\lg x = \log_2 x$
- **Logarithmic Functions:**

  $\log_a x = \Theta(\log_b x)$ for any $a, b > 1$

Polynomial Time

Reading: Sipser §7.2.

Time-bounded Simulations

- **Q:** How quickly can a 1-tape TM M_2 simulate a multitape TM M_1?
  - If M_1 uses $f(n)$ time, then it uses $\leq f(n)$ tape cells
  - M_2 simulates one step of M_1 by a complete sweep of its tape. This takes $\leq k \cdot f(n)$ steps.

  $\therefore$ M_2 uses $\leq f(n) \cdot \mathcal{O}(f(n)) = \mathcal{O}(f^2(n))$ steps in all.
- So $L \in \text{TIME}_{\text{multitape TM}}(f) \Rightarrow L \in \text{TIME}_{\text{1-tape TM}}(\mathcal{O}(f^2))$
- Similarly for
  - 2-D Tapes
  - Random Access TMs ...

Basic thesis of complexity theory

- **Extended Church-Turing Thesis:** Every "reasonable" model of computation can be simulated on a Turing machine with only a **polynomial** slowdown.

- Counterexamples?
  - Randomized computation.
  - Parallel computation.
  - Analog computers.
  - DNA computers.
  - Quantum computers.

Polynomial Time

- **Def:** Let $\text{P} = \bigcup_{p} \text{TIME}(p)$, where p is a polynomial
$$= \bigcup_{k \geq 0} \text{TIME}(n^k)$$
- P is also known as PTIME or $\mathcal{P}$
- *Coarse* approximation to "efficient algorithm"

Model-Independence of P

Although P is defined in terms of TM time, **P is a stable class, independent of the computational model.**

(Provided the model is reasonable)

Justification:

- If A and B are different models of computation,
  $L \in \text{TIME}_A(p_1(n))$,
  and B can simulate a time t computation of A in time $p_2(t)$,
  then $L \in \text{TIME}_B(p_2(p_1(n)))$.
- Polynomials are closed under composition, e.g.
  $f(n) = n^2, g(n) = n^3 + 1 \Rightarrow f(g(n)) = (n^3 + 1)^2 = n^6 + 2n^3 + 1.$

How much does representation matter?

- How big is the representation of an n-node directed graph?
  - . . . as a list of edges?
  - . . . as an adjacency matrix?
- How big is the representation of a natural number n?
  - . . . in binary?
  - . . . in decimal?
  - . . . in unary?

For which of the following do we know polynomial-time algorithms?

- Given a DFA M and a string w, decide whether M accepts w.
  - What is the "size" of a DFA?

- Given an NFA N, construct an equivalent DFA M.

More computational problems: are they in P?

- Given an NFA N and a string w, decide whether N accepts w.
- Given a regular expression R, construct an equivalent NFA N.
- Given a CFG G and a string w, decide whether G generates w.

And more computational problems: are they in P?

- Given two numbers n, m, compute their product.
  - What is the "size" of the numbers?
- Given a number n, decide if n is prime.
- Given a number n, compute n's prime factorization.

Another way of looking at P

- Multiplicative increases in time or computing power yield multiplicative increases in the size of problems that can be solved
- If L is in P, then there is a constant factor k such that
  - If you can solve problems of size s within a given amount of time
  - and you are given a computer that runs twice as fast, then
  - you can solve problems of size $k \cdot s$ on the new machine in the same amount of time.
- E.g. if L is decidable in $\mathcal{O}(n^d)$ time, then with twice as much time you can solve problems $2^{\frac{1}{d}}$ as large.

Exponential time

- $\mathsf{E} = \bigcup_{c>0} \mathsf{TIME}(c^n)$
- For problems in E, a multiplicative increase in computing power yields only an *additive* increase in the size of problems that can be solved.
- If L is in E, then there is a constant k such that
  - If you can solve problems of size s within a given amount of time
  - and you are given a computer that runs twice as fast, then
  - you can solve problems only of size $k + s$ on the new machine using the same amount of time.

NP

Reading: Sipser §7.3

"Nondeterministic Time"

- We say that a nondeterministic TM M **decides** a language L iff for every $w \in \Sigma^*$,
  1. Every computation by M on input w halts (in state q_{accept} or state q_{reject});
  2. $w \in L$ iff there exists at least one accepting computation by M on w.
  3. $w \notin L$ iff every computation by M on w rejects.
- M decides L in **nondeterministic time** $t(\cdot)$ iff for every w, every computation by M on w is of length at most $t(|w|)$

More on Nondeterministic Time

1. Linear speedup holds.
2. "Polynomial equivalence" holds among nondeterministic models

   e.g. L decided in time T by a nondeterministic multitape TM

   $\Rightarrow$ L decided in time $\mathcal{O}(T^2)$ by a nondeterministic 1-tape TM

Definition:

$\text{NTIME}(t(n)) =$
$\{L : L \text{ is decided in time } t(n) \text{ by some nondet. multitape TM}\}$

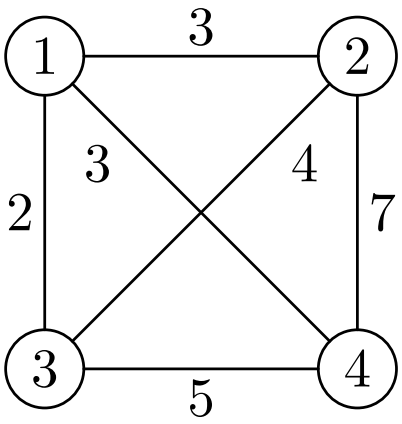
$$\text{NP} = \bigcup_{\text{polynomial}_p} \text{NTIME}(p) = \bigcup_{k \geq 0} \text{NTIME}(n^k).$$

P vs. NP

- Clearly P $\subseteq$ NP. But there are problems in NP that are **not obviously** in P ($\neq$ "obviously not")
- TSP $=$ TRAVELLING SALESMAN PROBLEM
  - Let $m > 0$ be the number of **cities**,
  - $D : \{1, \ldots, m\}^2 \to \mathcal{N}$ give the **distance** $D(i, j)$ between city i and city j, and
  - B be a distance **bound**

  Then TSP $= \{\langle m, D, B\rangle : \exists$ tour of all cities of length $\leq B\}$.

Traveling Salesman Problem: Example



$n = 4$

$B \geq 15 \Rightarrow \langle m, D, B \rangle \in \text{TSP}$
$B \leq 14 \Rightarrow \langle m, D, B \rangle \notin \text{TSP}$

"tour" = visits every city and returns to starting point

There are many variants of TSP, e.g. require visiting every city exactly once, triangle inequality on distances...

TSP $\in$ NP

- Why is TSP $\in$ NP?

  Because **if** $\langle m, D, B \rangle \in$ TSP, the following nondeterministic strategy will accept in time $\mathcal{O}(n^3)$, where $n =$ length of representation of $\langle m, D, B \rangle$.

  - **nondeterministically** write down a sequence of cities $c_1, \ldots, c_m$, for $m \leq n^2$. ("guess")
  - trace through that circuit and verify that the length is $\leq B$. If so, halt in q_{accept}. If not, halt in q_{reject}. (and "check")

  But any obvious **deterministic** version of this algorithm takes exponential time.

A useful characterization of NP

- A **verifier** for a language L is an algorithm V such that

$$L = \{x : V \text{ accepts } \langle x, y \rangle \text{ for some string } y\}.$$

- A **polynomial-time** verifier is one that runs in time polynomial in $|x|$ on input $\langle x, y \rangle$.
- A string y that makes $V(\langle x, y \rangle)$ accept is a "proof" or "certificate" that $x \in L$.
- **Example:** TSP

  certificate $y =$?

  $V(\langle x, y \rangle) =$?

- Without loss of generality, $|y|$ is at most polynomial in $|x|$.

NP is the class of easily verified languages

- **Theorem:** NP equals the class of languages with polynomial-time verifiers

  Proof:

  $\Rightarrow$

  $\Leftarrow$

- "L is in NP iff members of L have short, efficiently verifiable certificates"

NP is the class of easily verified languages

- **Theorem:** NP equals the class of languages with polynomial-time verifiers

  Proof:

  $\Rightarrow$ The y in $\langle x, y \rangle$ is a record of a computation that accepts x.

  $\Leftarrow$ Guess y and use the Verifier to check that V accepts $\langle x, y \rangle$

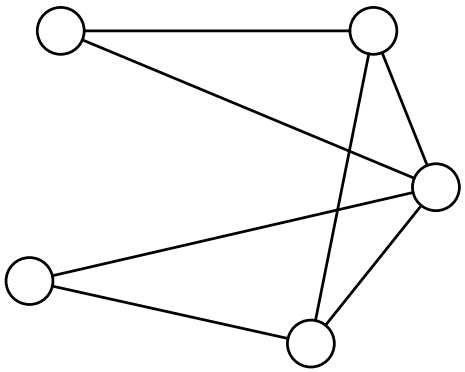
- "L is in NP iff members of L have short, efficiently verifiable certificates"

More problems in NP

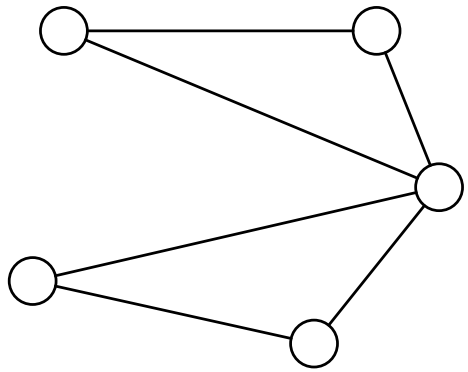
- HAMILTONIAN CIRCUIT

$$\text{HC} = \{G : G \text{ is an undirected graph with a circuit that touches each node just once}\}.$$

HC



No HC



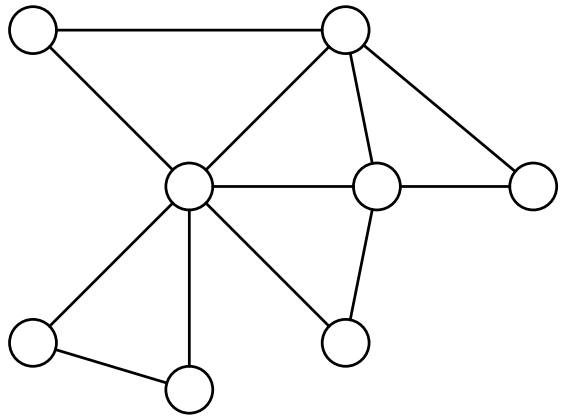
Really just a special case of TSP. (why?)

- We are not fussy about the precise method of representing a graph as a string, because all reasonable methods are within a polynomial of each other in length.

A “similar” problem that is in P

- EULERIAN CIRCUIT

$$\text{EC} = \{G : G \text{ is an undirected graph with a circuit that passes through each } \textit{edge} \text{ exactly once}\}.$$



It is easy to check if G is Eulerian...

So EC $\in$ P.

Composite Numbers

- COMPOSITES $= \{w : w$ is a composite number in binary$\}$.

  COMPOSITES $\in$ NP

  Not obviously in P, since an exhaustive search for factors would take at least proportional to the **value** of w, which grows as $2^n =$ exponential in the size of w.

  Only recently (2002), it was shown that COMPOSITES $\in$ P (equivalently, PRIMES $\in$ P).

Boolean logic

- Boolean formulas

  Def: A **Boolean formula** (B.F.) is either:

  - a "Boolean variable" $x, y, z, \ldots$
  - $(\alpha \vee \beta)$ where α, β are B.F.'s.
  - $(\alpha \wedge \beta)$ where α, β are B.F.'s
  - $\neg\alpha$ where α is a B.F.

  e.g. $(x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$

  [Omitting redundant parentheses]

Boolean satisfiability

Def: A **truth-assignment** is a mapping
$\mathcal{A}:$ Boolean variables $\to \{0, 1\}$. [$0 =$ false, $1 =$ true]

A T-A is extended to all B.F.'s by the rules:

- $\mathcal{A}(\alpha \vee \beta) = 1$ iff $\mathcal{A}(\alpha) = 1$ or $\mathcal{A}(\beta) = 1$
- $\mathcal{A}(\alpha \wedge \beta) = 1$ iff $\mathcal{A}(\alpha) = 1$ and $\mathcal{A}(\beta) = 1$
- $\mathcal{A}(\neg\alpha) = 1$ iff $\mathcal{A}(\alpha) = 0$

$\mathcal{A}$ **satisifies** α (written $\mathcal{A} \models \alpha$) iff $\mathcal{A}(\alpha) = 1$

In this case, α is **satisfiable**. If no $\mathcal{A}$ satisfies α, then α is **unsatisfiable**.

SAT $= \{\alpha : \alpha$ is a satisfiable Boolean formula$\}$.

Prop: SAT $\in$ NP

A "similar" problem in P: 2-SAT

A 2-CNF formula is one that looks like

$$(x \vee y) \wedge (\neg y \vee z) \wedge (\neg y \vee \neg x)$$

i.e., a conjunction of **clauses**, each of which is the disjunction of 2 literals (or 1 literal, since $(x) \equiv (x \vee x)$)

2-SAT $=$ the set of satisfiable 2-CNF formulas.

e.g. $(x \vee y) \wedge (\neg x \vee \neg y) \wedge (\neg x \vee y) \wedge (x \vee \neg y) \notin$ SAT

2-SAT $\in$ P

Method (resolution):

1. If x and $\neg x$ are both clauses, then *not* satisfiable

   e.g. $(x) \wedge (z \vee y) \wedge (\neg x)$

2. If $(x \vee y) \wedge (\neg y \vee z)$ are both clauses, add clause $(x \vee z)$ (which is implied).

3. Repeat. If no contradiction emerges $\Rightarrow$ satisfiable.

$\mathcal{O}(n^2)$ repetitions of step 2 since only 2 literals/clause.

P vs. NP

- We would like to solve problems in NP efficiently
- We know P $\subseteq$ NP.
- Problems in P can be solved “fairly” quickly.
- What is the relationship between P and NP?

NP and Exponential Time

Claim: $\text{NP} \subseteq \bigcup_k \text{TIME}(2^{n^k})$

Of course, this gets us nowhere near P.

Is $\text{P} = \text{NP}$?

i.e., do all the NP problems have polynomial time algorithms?

It doesn't "feel" that way but as of today there is no NP problem that has been **proven** to **require** exponential time!

The Strange, Strange World if P = NP

- Thousands of important languages can be decided in polynomial time, e.g.
  - SATISFIABILITY
  - TRAVELING SALESMAN
  - HAMILTONIAN CIRCUIT
  - MAP COLORING
  - ⋮

If P = NP, then searching becomes easy

- Every "reasonable" search problem could be solved in polynomial time.
  - "reasonable" $\equiv$ solutions can be recognized in polynomial time (and are of polynomial length)
  - SAT SEARCH: Given a satisfiable boolean formula, find a satisfying assignment.
  - FACTORING: Given a natural number (in binary), find its prime factorization.
  - NASH EQUILIBRIUM: Given a two-player "game", find a Nash equilibrium.
  - $\vdots$

If P = NP, Optimization becomes easy

- Every "reasonable" optimization problem can be solved in polynomial time.
  - Optimization problem $\equiv$ "maximize (or minimize) $f(x)$ subject to certain constraints on x"
  - "Reasonable" $\equiv$ "f and constraints are poly-time"
  - MIN-TSP: Given a TSP instance, find the shortest tour.
  - SCHEDULING: Given a list of assembly-line tasks and dependencies, find the maximum-throughput scheduling.
  - PROTEIN FOLDING: Given a protein, find the minimum-energy folding.
  - CIRCUIT MINIMIZATION: Given a digital circuit, find the smallest equivalent circuit.

If P = NP, Secure Cryptography becomes impossible

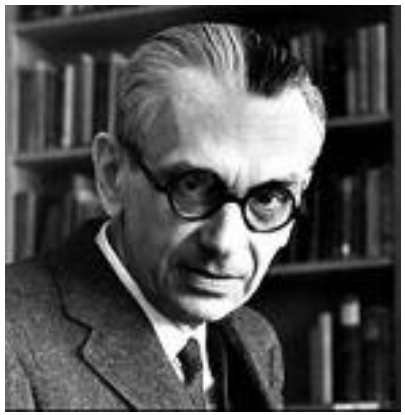
- **Cryptography:** Every encryption algorithm can be "broken" in polynomial time.
  - "Given an encryption z, find the corresponding decryption key K and message m" is an NP search problem.

If P = NP, Artificial Intelligence becomes easy

- **Artificial Intelligence:** "Learning" is easy.
  - Given many examples of some concept (e.g. pairs (image1, "dog"), (image2, "person"), . . .), classify new examples correctly.
  - Turns out to be equivalent to finding a short "classification rule" consistent with examples.

If P = NP, Even Mathematics Becomes Easy!

- **Mathematical Proofs:** Can always be found in polynomial time (in their length).
  - SHORT PROOF: Given a mathematical statement S and a number n (in unary), decide if S has a proof of length at most n (and, if so, find one).
  - An NP problem!
  - cf. letter from Gödel to von Neumman, 1956.



Gödel's Letter to von Neumann, 50 years ago

[$\phi(n) =$ time required for a TM to determine whether a formula has a proof of length n]

...

If there really were a machine with $\phi(n) \sim k \cdot n$ (or even $\sim k \cdot n^2$) this would have consequences of the greatest importance. Namely, it would obviously mean that in spite of the undecidability of the Entscheidungsproblem, the mental work of a mathematician concerning Yes-or-No questions could be completely replaced by a machine. ...

It would be interesting to know, for instance, the situation concerning the determination of primality of a number and how strongly in general the number of steps in finite combinatorial problems can be reduced with respect to simple exhaustive search. ...

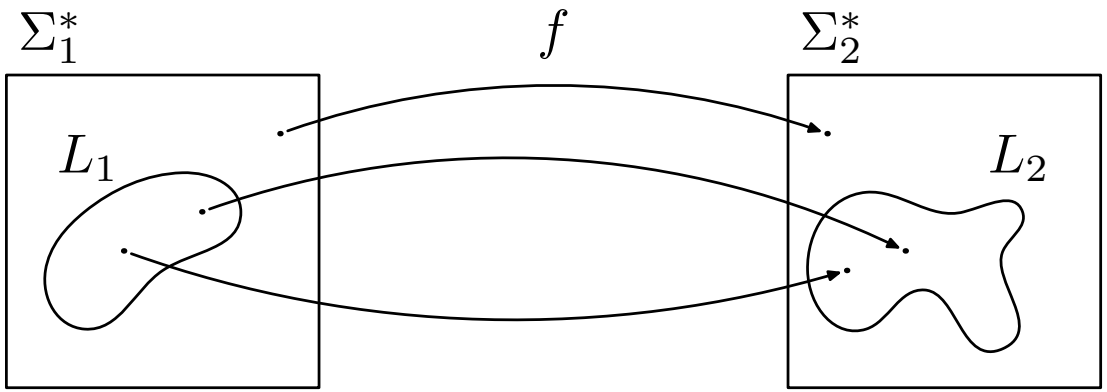
The World if P $\neq$ NP?

- **Q:** If P $\neq$ NP, can we conclude anything about any *specific* problems?
- **Idea:** Try to find the "hardest" NP language.
  - Just like A_{TM} was the "hardest" Turing-recognizable language.
  - Want $L \in$ NP such that $L \in$ P iff **every** NP language is in P.

Polynomial-time reducibility

- **Def:** $L_1 \leq_{\mathsf{P}} L_2$ iff there is a **polynomial-time** computable function $f : \Sigma_1^* \to \Sigma_2^*$ such that for every $x \in \Sigma_1^*$, $x \in L_1$ iff $f(x) \in L_2$.
- **Proposition:** If $L_1 \leq_{\mathsf{P}} L_2$ and $L_2 \in \mathsf{P}$, then $L_1 \in \mathsf{P}$.
- **Proof:**

$L_1 \leq_{\mathsf{P}} L_2$



$x \in L_1 \Rightarrow f(x) \in L_2$

$x \notin L_1 \Rightarrow f(x) \notin L_2$

f computable in polynomial time

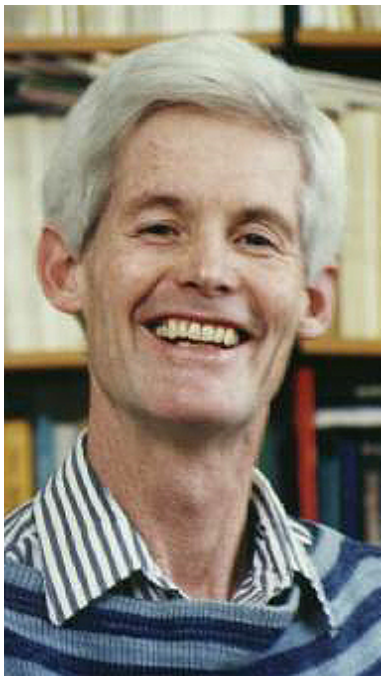
$L_2 \in \mathsf{P} \Rightarrow L_1 \in \mathsf{P}$.

NP-Completeness

- **Def:** L is **NP-complete** iff
  1. $L \in$ NP and
  2. Every language in NP is reducible to L in polynomial time. ("L is **NP-hard**")
- **Prop:** Let L be any NP-complete language. Then P $=$ NP *if and only if* $L \in$ P.

Cook-Levin Theorem

- Stephen Cook 1971, Leonid Levin 1973
- **Theorem:** SAT (Boolean satisfiability) is NP-complete
- **Proof:** Need to show that **every** language in NP reduces to SAT (!) Proof later.



NP-Completeness

- **Reading**: Sipser §7.4, §7.5.
- For "culture": *Computers and Intractability: A Guide to the Theory of NP-completeness*, by Garey & Johnson.

More NP-complete problems

From now on we prove NP-completeness using:

Lemma: If we have the following

- L is in NP
- $L_0 \leq_{\mathsf{P}} L$ for some NP-complete L_0

Then L is NP-complete

Proof:

More NP-complete problems

From now on we prove NP-completeness using:

Lemma: If we have the following

- L is in NP
- $L_0 \leq_P L$ for some NP-complete L_0

Then L is NP-complete

Proof: Since by hypothesis $L \in$ NP, it suffices to show that every $L' \in$ NP reduces to L.

- $L' \leq_P L_0$ since L_0 is NP-complete;
- $L_0 \leq_P L$ by hypothesis; and so
- $L' \leq_P L$ by transitivity.

Thus, L is NP-complete.

3-SAT

Def: A Boolean formula is in **3-CNF** if it is of the form:

$$C_1 \wedge C_2 \wedge \ldots \wedge C_n$$

where each clause C_i is a disjunction ("or") of 3 literals:

$$C_i = (C_{i1} \vee C_{i2} \vee C_{i3})$$

where each literal C_{ij} is either

- a variable x, or
- the negation of a variable, $\neg x$.

e.g. $(x \vee y \vee z) \wedge (\neg x \vee \neg u \vee w) \wedge (u \vee u \vee u)$

3-SAT is the set of **satisfiable** 3-CNF formulas.

3-SAT is NP-complete

Proof: Show that SAT $\leq_P$ 3-SAT.

1. Given an arbitrary Boolean formula, e.g.

$$F = (\neg((x \vee \neg y) \wedge (z \vee w)) \vee \neg x).$$
$$\quad 1 \quad 2\ 3 \quad 4 \quad 5 \quad 6\ 7$$

2. Number the operators.
3. Select a new variable a_i for each operator.
   The variable a_i is supposed to mean "the subformula rooted at operator i is true."
4. Write a formula stating the relation between each subformula and its children subformulas.

Reduction of SAT to 3-SAT, continued

For example, where

$$F = (\underset{1}{\neg}(\underset{2}{(}x \vee \underset{3}{\neg} y\underset{4}{)} \wedge \underset{5}{(}z \vee w\underset{6}{)}) \vee \underset{7}{\neg} x),$$

$$F_1 = \left(\begin{array}{lll} & (a_3 \equiv \neg y) & \wedge \quad (a_7 \equiv \neg x) \\ \wedge & (a_2 \equiv x \vee a_3) & \wedge \quad (a_1 \equiv \neg a_4) \\ \wedge & (a_5 \equiv z \vee w) & \wedge \quad (a_6 \equiv a_1 \vee a_7) \\ \wedge & (a_4 \equiv a_2 \wedge a_5) & \end{array} \right)$$

5. Let k be the number of the main operator/subformula of F. (Note: $k = 6$ in the example)

Write F_1 in 3-CNF to obtain F_2

- $\alpha \equiv \beta$ is the same as $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$, so to get rid of the $\alpha \equiv \beta$ clauses in F_1,

Replace	By
$a \equiv \neg b$	$(a \vee a \vee b) \wedge (\neg a \vee \neg a \vee \neg b)$
$a \equiv b \vee c$	$(\neg a \vee b \vee c) \wedge (\neg b \vee \neg b \vee a) \wedge (\neg c \vee \neg c \vee a)$
$a \equiv b \wedge c$	$(\neg a \vee \neg a \vee b) \wedge (\neg a \vee \neg a \vee c) \wedge (\neg b \vee \neg c \vee a)$

Output of the reduction: $a_k \wedge F_2$.

Make sure this reduction is polynomial-time

Q: Does this prove that every Boolean formula can be converted to 3-CNF?

- Note that transforming boolean formulas to CNF by "multiplying out" does *not* show that the satisfiable CNF formulas are an NP-complete set.

  (Much less that 3-SAT is NP-complete.)

  Non-polynomial transformation:

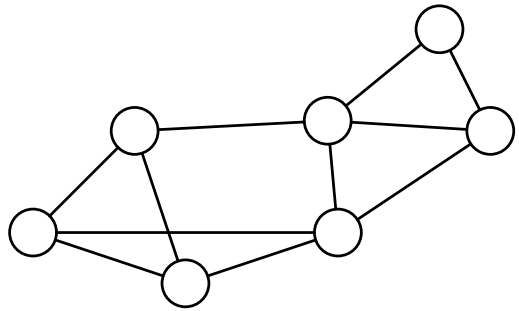
  $x_1 y_1 \vee x_2 y_2 \vee \ldots \vee x_n y_n$ (which has size $\mathcal{O}(n)$) becomes

  $(x_1 \vee \ldots \vee x_n) \wedge (x_1 \vee \ldots \vee x_{n-1} \vee y_n) \wedge \ldots \wedge (y_1 \vee \ldots \vee y_n)$

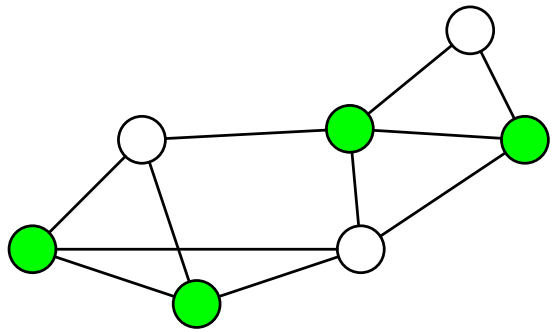
  (size $\mathcal{O}(2^n)$)

VERTEX COVER (VC)

- **Instance:**
  - a graph
  - a number k, (e.g. 4)



- **Question:** Is there a set of k vertices that "cover" the graph, i.e., include at least one endpoint of every edge?



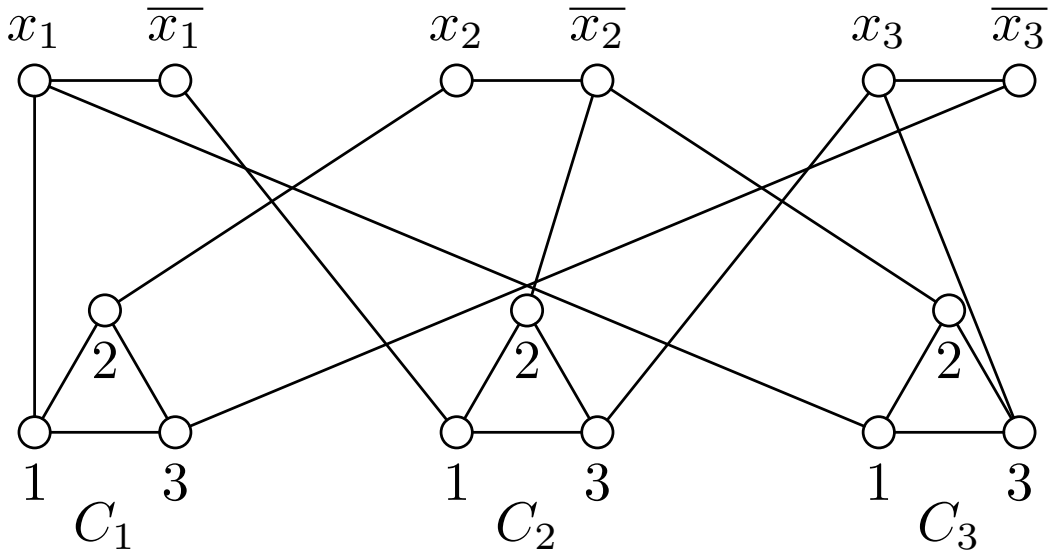
VC is NP-complete

- VC is in NP:
- 3-SAT $\leq_P$ VC:
  - Let F be a 3-CNF formula with clauses $C_1, \ldots, C_m$, variables $x_1, \ldots, x_n$.
  - We construct a graph G_F and a number N_F such that:

    G_F **has a size** N_F **vertex cover iff** F **is satisfiable**

Construction of G_F and N_F from F

E.g. $F = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3)$



- $G_F =$ one dumbbell for each variable, one triangle for each clause, and corner j of triangle i is connected to the vertex representing the jth literal in C_i.
- $N_F = 2m + n = 2(\#\text{ clauses}) + (\#\text{ variables})$.
  $\Rightarrow$ 1 vertex from each dumbbell and 2 from each triangle.

If F is satisfiable then there is an N_F cover

Proof: Choose a node from each dumbbell according to the truth-assignment.

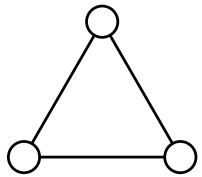
Choose the other 2 corners from each triangle.

If G_F has a N_F cover then F is satisfiable

Proof: The cover must include

- At least one vertex from each

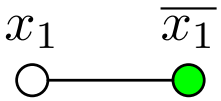
- At least two vertices from each



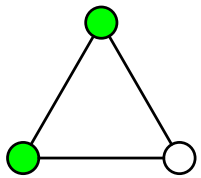
C_1

But this totals N_F, and so there must be

- **Exactly** one vertex from each

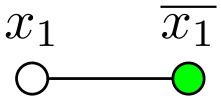


- **Exactly** two vertices from each



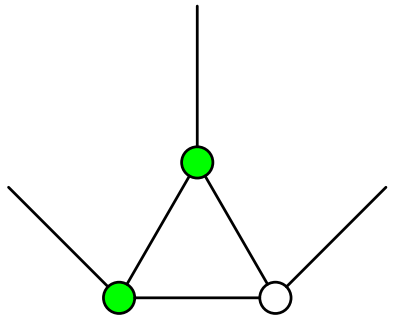
C_1

Getting a satisfying truth-assignment from a vertex cover

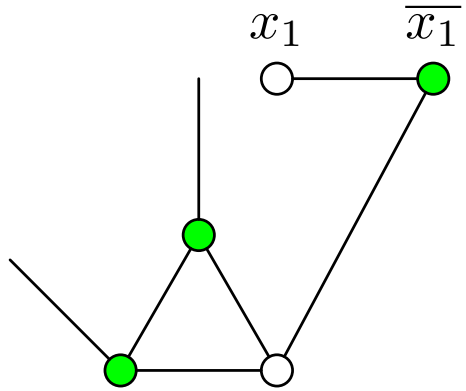
The selections from the 

are a truth-assignment.

Of the 3 “cross-edges” from each triangle:



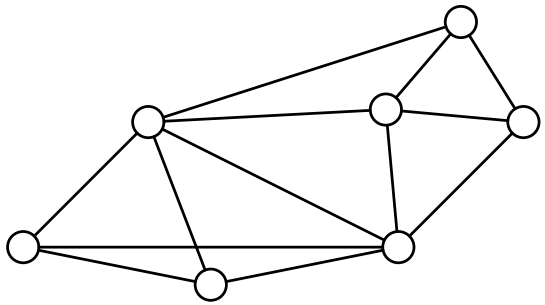
Exactly 2 are covered by the choices from the triangle, so one must be covered by the choice from the dumbbell:



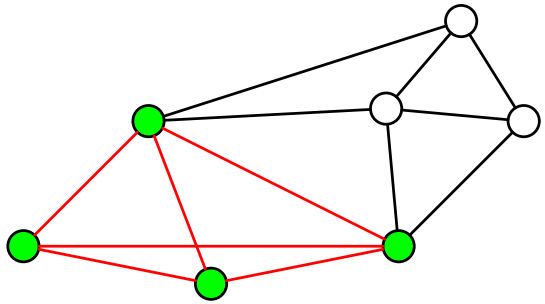
This means that the truth-assignment from the dumbbell satisfies at least one literal from each clause.

CLIQUE

- **Instance:**
  - a graph, e.g.



  - a number k (e.g. 4)
- **Question:** Is there a clique of size k, i.e., a set of k vertices such that there is an edge between each pair?



- Easy to see that CLIQUE $\in$ NP.

VC $\leq_P$ CLIQUE

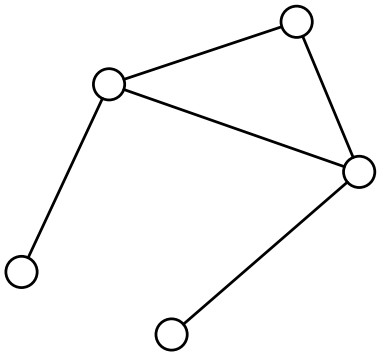
If G is any graph, let G^c be the graph with the same vertices such that:

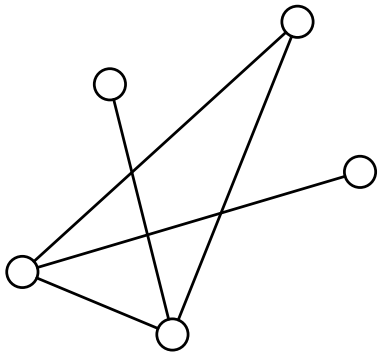
there is an edge between x and y in G^c

iff

there is **no** edge between x and y in G

e.g.

$G =$ 

$G^c =$ 

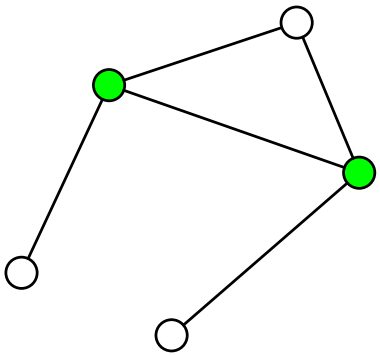
VC $\leq_P$ CLIQUE, continued

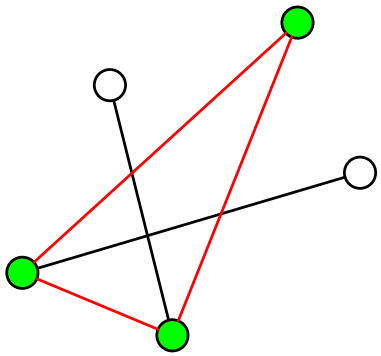
Let (G, k) be an instance of VC.

Claim: G has a k-cover iff G^c has a $|G| - k$ clique, where $|G|$ is the number of vertices in G.

(So the mapping $(G, k) \mapsto (G^c, |G| - k)$ is a reduction of VC to CLIQUE.)

Proof (by example):

$G =$ 

$G^c =$ 

INTEGER LINEAR PROGRAMMING

An **integer linear program** is

- A set of variables $x_1, \ldots, x_n$ which must take integer values.
- A set of linear inequalities:

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq c_i \qquad [i = 1, \ldots, m]$$

e.g. $x_1 - 2x_2 + x_4 \leq 7$

$x_1 \geq 0 \qquad [-x_1 \leq 0]$

$x_4 + x_1 \leq 3$

ILP $=$ the set of integer linear programs for which there are values for the variables that simultaneously satisfy all the inequalities.

ILP is NP-complete

INTEGER LINEAR PROGRAMMING $\in$ NP. (Not obvious! Need a little math to prove it. Proof omitted.)

INTEGER LINEAR PROGRAMMING is NP-hard: by reduction from 3-SAT (3-SAT $\leq_P$ ILP). Given 3-CNF Formula F, construct following ILP P as follows:

Example of 3-SAT $\leq_P$ INTEGER LINEAR PROGRAMMING

- Suppose
  $F = (x \vee y \vee \neg z) \wedge (\neg x \vee \neg y \vee w) \wedge (\neg y \vee \neg w \vee \neg z) \wedge (x \vee w \vee z)$

  Then ILP P can be:

$x \geq 0 \quad x \leq 1$	$y \geq 0 \quad y \leq 1$	etc.
$x + \neg x = 1$	$y + \neg y = 1$	etc.
$x + y + \neg z \geq 1$	$\neg x + \neg y + w \geq 1$	
$\neg y + \neg w + \neg z \geq 1$	$x + w + z \geq 1$	

  The integer variables are $x, y, z, w, \neg x, \neg y, \neg z, \neg w$.

  F is satisfiable iff

  P can be satisfied (with values of variables $\in \{0, 1\}$)

  Note: LINEAR PROGRAMMING where the variables can take *real* values is known to be in P.

More NP-complete/NP-hard Problems

- HAMILTONIAN CIRCUIT (and hence TSP) (see Sipser for related problems)
- SCHEDULING
- CIRCUIT MINIMIZATION
- SHORT PROOF
- NASH EQUILIBRIUM WITH MAXIMUM PAYOFF
- PROTEIN FOLDING
- ⋮
- See Garey & Johnson for hundreds more.

Cook-Levin Theorem and Beyond

Reading: (none)

Cook-Levin Theorem: SAT is NP-complete

Proof:

- Already know SAT $\in$ NP, so only need to show SAT is NP-hard.
- Let L be any language in NP. Let M be a NTM that decides L in time n^k.

  We define a polynomial-time reduction

  $$f_L : \text{inputs} \mapsto \text{formulas}$$

  such that for every w,

  $$M \text{ accepts input } w \text{ iff } f_L(w) \text{ is satisfiable}$$

Reduction via "computation histories"

Proof idea: satisfying assignments of $f_L(w) \leftrightarrow$ accepting computations of M on w

Describing computations of M by boolean variables:

- If $n = |w|$, then any computation of M on w has at most n^k configurations.
- Each configuration is an element of C^{n^k}, where $C = Q \cup \Gamma \cup \{\#\}$ (mark left and right ends with $\{\#\}$).

$\rightsquigarrow$ computation depicted by $n^k \times n^k$ "tableau" of members of C.

- Represent contents of cell (i, j) by $|C|$ boolean variables $\{x_{i,j,s} : s \in C\}$, where $x_{i,j,s} = 1$ means "cell (i, j) contains s".
- $0 \leq i, j \leq n^k$, so $|C| \cdot n^{2k}$ boolean variables in all

Subformulas that verify the computation

Express conditions for an accepting computation on w by boolean formulas:

- $\phi_{\text{cell}} =$
  "for each (i, j), there is exactly one $s \in C$ such that $x_{i,j,s} = 1$".
- $\phi_{\text{start}} =$ "first row equals start configuration on w"
- $\phi_{\text{accept}} =$ "last row is an accept configuration on w"
- $\phi_{\text{move}} =$ "every 2×3 window is consistent with the transition function of M"

Completing the proof

Claim: Each of above can be expressed by a formula of size $\mathcal{O}((n^k)^2) = \mathcal{O}(n^{2k})$, and can be constructed in polynomial time from w.

Claim: M has an accepting computation on w if and only if $f_L(w) = \phi_{\text{cell}} \wedge \phi_{\text{start}} \wedge \phi_{\text{accept}} \wedge \phi_{\text{move}}$ has a satisfying assignment.

Thus $w \mapsto f_L(w)$ is a polynomial-time reduction from L to SAT.

Since above holds for every $L \in$ NP, SAT is NP-hard, as desired. ■

Glimpses Beyond: co-NP

Recall that co-NP $= \{\overline{L} : L \in \text{NP}\}$.

Some co-NP-complete problems:

- Complement of any NP-complete problem.
- TAUTOLOGY = $\{\varphi : \forall a\ \varphi(a) = 1\}$ (even for 3-DNF formulas φ).

Believed that NP $\neq$ co-NP, P $\neq$ NP $\cap$ co-NP.

"Between" P and NP-complete

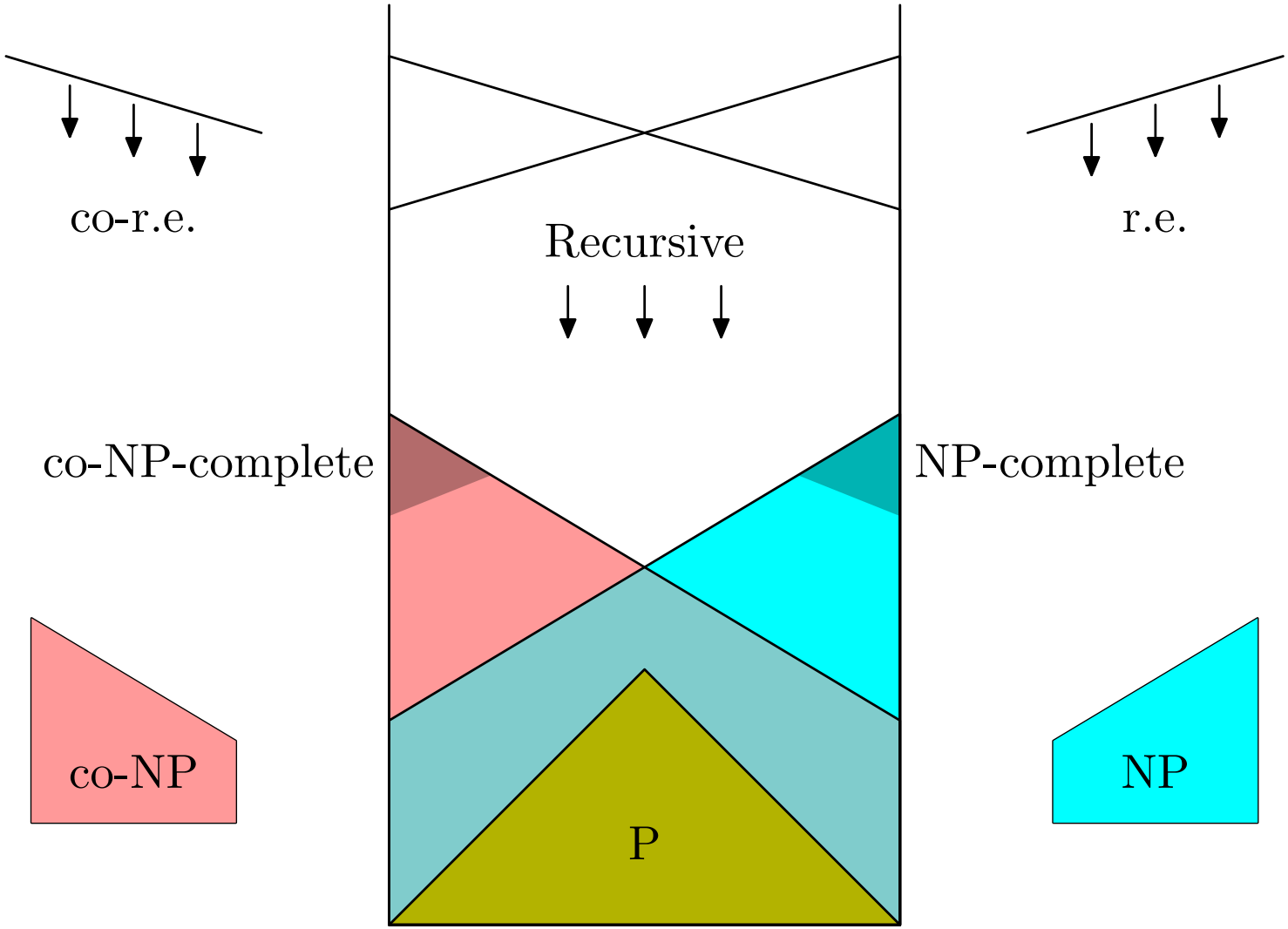
Theorem: If P $\neq$ NP, then there are NP languages that are neither in P nor NP-complete.

Some natural candidates:

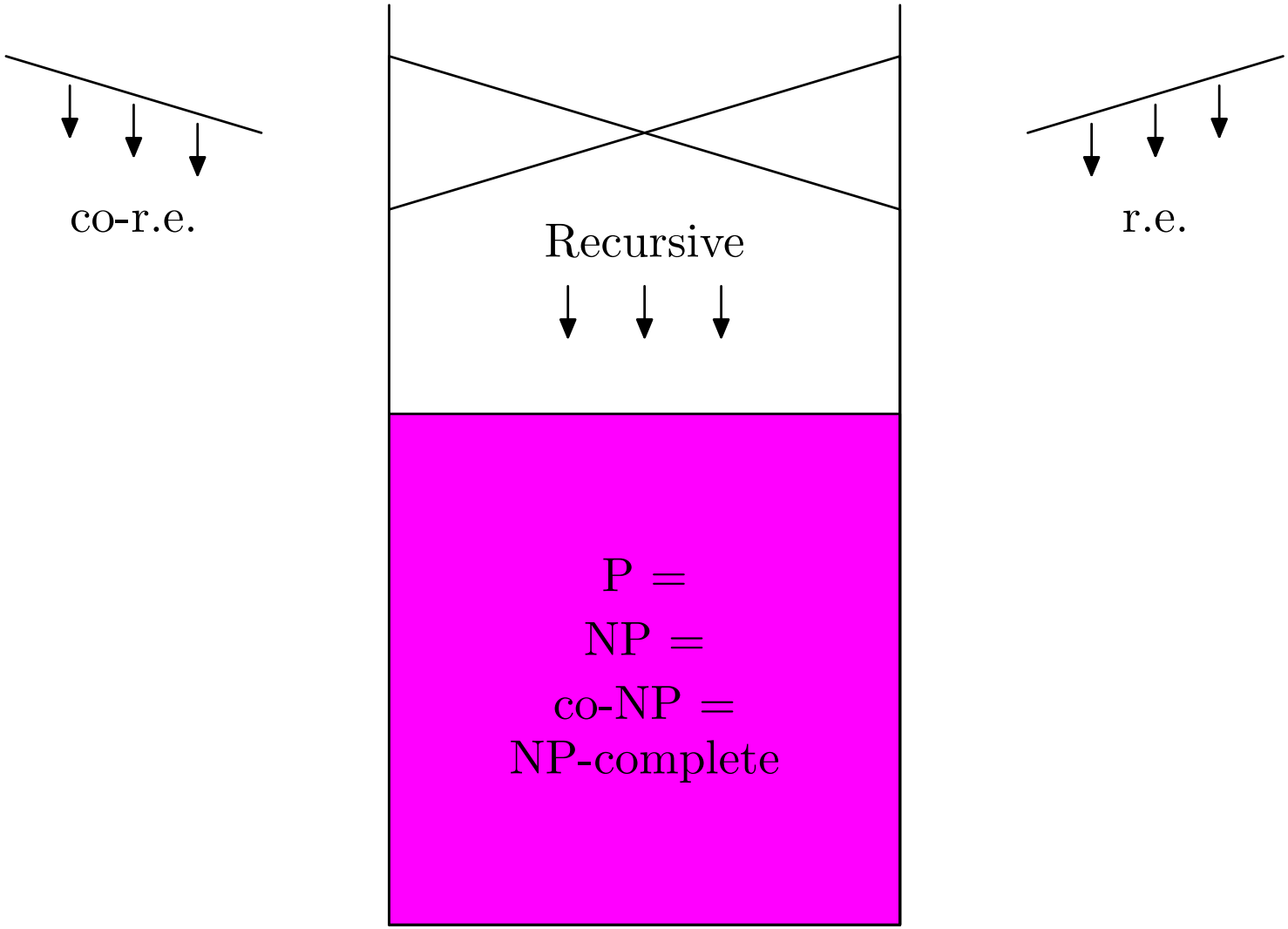
- FACTORING (when described as a language)
- NASH EQUILIBRIUM
- GRAPH ISOMORPHISM
- Any problem in NP $\cap$ co-NP for which we don't know a poly-time algorithm.

Despite decades of effort, we don't know ANY nontrivial lower bound on the complexity of any NP-complete problem!

The World If P $\neq$ NP



The World If P = NP



Beyond NP

- "Space" as a resource = number of TM squares used in a computation
- A reasonable proxy for memory on other computational models
- PSPACE = languages decidable on TMs using polynomial space
- P $\subseteq$ PSPACE (why?)
- NP $\subseteq$ PSPACE (why?)
- PSPACE $\subseteq$ NPSPACE (why?)

Examples of problems in PSPACE or NPSPACE but probably not in NP

- Determining whether $w \in L(G)$, where G is a context-sensitive grammar, is in NPSPACE
  - Recall that G is a CSG if the right-hand side of every rule is at least as long as its left-hand side
  - I.e. if $u \to v$ is a rule then $|u| \leq |v|$
- Determining whether a quantified boolean expression is true is in PSPACE
  - E.g. $\forall x \exists y \forall z[(x \wedge y) \vee (\neg x \wedge \neg z)]$

Savitch's Theorem

NPSPACE $=$ PSPACE

- How much time can a computation take if it uses $\mathcal{O}(n^k)$ space and does not loop? $\mathcal{O}(2^{n^k})$
- To check deterministically if there exists a computation from TM configuration C_1 to TM configuration C_2 in T steps,
  - for all configurations C, check if
    - there is a computation from C_1 to C of $T/2$ steps and
    - there is a computation from C to C_2 in $T/2$ steps
- To check whether the initial configuration yields the final configuration takes $\mathcal{O}(\log(2^{n^k})) = \mathcal{O}(n^k)$ recursion levels and $\mathcal{O}(n^k)$ space at each level $\Rightarrow \mathcal{O}(n^{2k})$ space

Conclusions

Reading: (none)

Reprise: Models of computation and formal systems

- DFAs, NFAs, REs, CFGs, PDAs, TMs, NTMs, ...
- How to formally **model computation**
- **Asymptotic perspective** (fixed program for all input lengths)
- Design your own models as circumstances demand (e.g. interactive/distributed computation, randomized computation, biological systems, economic systems)

Classification of computational problems

- **Positive results:** regular, context-free, polynomial-time, decidable, Turing-recognizable languages
- **Negative results:** non-regular, non-CF, NP-complete(?), undecidable, non-recognizable languages
- Notion of **reduction** between problems
- The systematic methodology for proving things impossible is one of the most important achievements of computer science
- NP-completeness is one of the most important "exports" of computer science to the rest of science

Understanding Intractability

- Many important problems are NP-complete (or even undecidable)
- But also some great positive results in algorithms design
  - E.g. poly-time algorithms for LINEAR PROGRAMMING, PRIMALITY TESTING, POLYNOMIAL FACTORIZATION, NETWORK FLOWS, ...
- What does NP-completeness mean? (assuming P $\neq$ NP)
  - No algorithm can be guaranteed to solve the problem perfectly in polynomial time on **all** instances
  - Exhaustive search is often unavoidable
  - Mathematical nastiness: no nice, closed form solutions

Coping with Intractability

- What if you need to solve an NP-complete (or undecidable) problem?
  - Ask your boss for a new assignment. :-)
  - Simplify the problem, you may not need to solve it in full generality
  - Identify **additional constraints** that make the problem easier (e.g. bounded-degree graphs, ILP with fixed number of variables, 2-SAT)
  - **Approximation algorithms**, e.g., find a TSP tour of length at most 1.01 times the shortest
  - **Average-case analysis**—analyse running time or correctness on "random" inputs. (Often hard to find distribution that models "real-life" inputs well.)

More attacks on intractable problems

- **Heuristics**—techniques that seem to work well in practice but do not have rigorous performance guarantees.
- **Change the problem**
  - Instead of verifying that general programs satisfy desired security properties (undecidable), ask programmers to supply programs with (easily verifiable) “proofs” that the properties are satisfied
  - Change the programming language